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Diagnosability Analysis of Multi-Station Manufacturing Processes

Variation propagation in a multi-station manufacturing process (MMP) is described by the theory of "Stream of Variation." Given that the measurements are obtained via certain sensor distribution scheme, the problem of whether the stream of variation of an MMP is diagnosable is of great interest to both academia and industry. We present a comprehensive study of the diagnosability of MMPs in this paper. It is based on the state space model and is parallel to the concept of observability in control theory. Analogous to the observability matrix and index, the diagnosability matrix and index are first defined and then derived for MMP systems. The result of diagnosability study is applied to the evaluation of sensor distribution strategy. It can also be used as the basis to develop an optimal sensor distribution algorithm. An example of a three-station assembly process with multi-fixture layouts is presented to illustrate the methodology. [DOI: 10.1115/1.1435645]

1 Introduction

1.1 Problem Statement. A multi-station manufacturing process (MMP) can be defined as a process involving operations on multiple work stations to manufacture a product. Examples of MMPs include: 1) the automotive body assembly, in which multiple parts are assembled on multiple stations; 2) the transfer-line machining process, which involves multiple machining operations of a single part on multiple stations; and 3) the progressive die stamping process, which involves multiple stamping stations to form one part. Statistical analysis of variation propagation in MMPs is important to quality and productivity improvement.

A product inspection-oriented measurement strategy is usually employed in industrial practice to ensure product quality. Following this strategy, measurements are taken directly from the finished and/or intermediate product, and then compare with the product design nominals. The measurement selection is solely based on product features and their specifications. If the measurements show that there are some product features out of their specifications (e.g., there is an excessive deviation from the design nominal or large variability), this measurement strategy alone may not lead to the identification of the root causes of quality defects. The problem of root cause identification is very challenging, especially for MMPs, due to the large amount of needed information and complex variation propagation during the manufacturing process.

As shown in Fig. 1, the quality information flow in an N-station process can be arranged in a parallelepiped with each cross-section representing a work station. On each station, there are data of multiple attributes (denoted as M_1, $M_2, \ldots,$ and M_m) that are produced continuously as production proceeds. For instance, curves shown in the final cross-section in Fig. 1 are representations of time series data obtained during production. Generally, there exists an autocorrelation (in terms of time) among data of the same attribute and a cross-correlation among data of different attributes. On the other hand, every attribute can be tracked on all stations. For example, the thick line demonstrated in Fig. 1 represents the tracking of attribute M_2. More complicated relationship exists in the autocorrelation of each attribute on different stations. The quality information flow in the parallelepiped is analogous to the water flow in a river. This analogy leads to the term "Stream of Variation (SOV)" used to describe the propagation of quality information in an MMP [1].

It is very common that manufacturers take measurements of a large sample of products on a given station, which only requires to install sensors on that specific station. The information obtained in this way is equivalent to the data produced within a *y-z* cross-section of the parallelepiped in Fig. 1. Thus, the data are referred to as "*cross-sectional*" data in literature [2]. The question of interest is whether the stream of variation of an MMP is diagnosable, i.e., if the root causes of quality-related problem can be identified, given the sensor distribution at selected "cross-sections." In this paper, we address the diagnosability of MMPs in terms of sensor distribution and process configuration in a general setting.

1.2 Related Work. In a manufacturing system, diagnosability could be given different focus in different situations. Usually, it is classified as fault detectability and distinguishability. Detectability measures the performance of recognizing fault occurrence while distinguishability refers to the capability of identifying the root cause of an occurred fault. In an MMP, the occurrence of faults can be detected by using SPC (Statistical Process Control) control charts [3], which are constructed based on measurements of the final product. However, the SPC control charts cannot identify the root causes of specific quality defects. In fact, the diagnosability that we are concerned about corresponds to fault distinguishability, namely the ability to identify root cause. When SPC charts are used, it is the operators/engineers' job, to identify what the root cause is, based on their experience. It is then difficult to identify root cause in an MMP since quality defects that are detected on the current station could be caused by variation transmission and accumulation from upstream stations. One straightforward solution is to apply the SPC techniques to every single station in an MMP since the root cause within a single station is easier to identify based on operators' empirical knowledge. The implementation of this solution is neither necessary nor economical since it involves obtaining measurement, constructing controller, and implementing control charts at each station. Actually this solution overlooks the inherent relationship among different stations. In this case, a vast amount of useful information is not fully utilized to reduce the number of sensors, actuators, and control charts.

An innovative approach is to establish analogy between station index and time index so that the quality information in the MMP can be put into a serial order, which looks like "time" series data (e.g., the thick line in Fig. 1). This analogy functions as a mecha-

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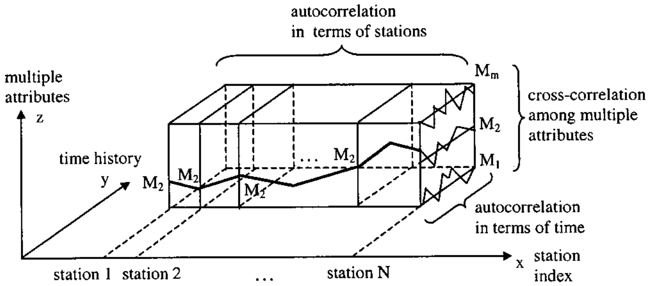

Fig. 1 Information flow in multi-station manufacturing processes

nism to link together information at different stations. But Shewhart charts are not appropriate for directly monitoring such kind of "time series" data because the quality values of product features on a downstream station are affected by those on upstream stations, and thus the data are likely to be correlated among stations. Control theory fits better to analyze these autocorrelated data (in terms of station) generated through the MMP if appropriate stream-of-variation models could be developed to characterize the quality information in an MMP, as opposed to the traditional models that describe the behavior of a single manufacturing station [4–7].

Several SOV models were proposed recently such as *state transition model* [8], *AR(1) model* [2,9], and *state space model* [10,11]. Mantripragada and Whitney [8] adopted the concept of output controllability from control theory to evaluate and improve the automotive body structure design in order to reduce the dimensional variation. Lawless et al. [2,9] investigated variation transmission in both assembly and machining processes by estimating parameters in their AR(1) model with the measured quality information.

When the MMP models are used for diagnosis, the diagnosability is an important issue to be investigated together with the development of the diagnosis algorithm. In [12], the diagnosability is defined in the following way.

If a system can be modeled by a linear input-output relationship as

$$\mathbf{y}_t = \mathbf{C} \cdot \mathbf{x}_t + \boldsymbol{\varepsilon}_t \tag{1}$$

where $\mathbf{y}_t$ is the measurement vector, $\mathbf{x}_t$ is the state vector to be estimated, and $\boldsymbol{\varepsilon}_t$ is the sensor noise, then the system diagnosability is satisfied if $\mathbf{C}$ is of full rank. In the situation that the input-output relationship is nonlinear, linearization is conducted at some setpoints and the higher order uncertainties are merged with $\boldsymbol{\varepsilon}_t$ as the new noise term. In dynamic systems, the diagnosability is equivalent to the observability of state variables. Consider a time-invariant linear system. The state vector $\mathbf{x}_t$ is governed by the following dynamic equation

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \boldsymbol{\xi}_t \tag{2}$$

where $\mathbf{u}_t$ is the input vector, $\boldsymbol{\xi}_t$ is the process background noise, and $\mathbf{A}$ is the dynamic matrix. The observability grammian $\mathbf{W}_0$ is defined as the solution to a Lyapunov equation [13]

$$\mathbf{A}^T\mathbf{W}_0 + \mathbf{W}_0\mathbf{A} = -\mathbf{C}^T\mathbf{C} \tag{3}$$

Thus, this dynamic system is said to be observable if $\mathbf{W}_0$ is of full rank.

In the two aforementioned approaches (state transition model and AR(1) model), measurements are assumed to be available at every station in the process. With that assumption, the MMP is always diagnosable. Both authors suggested the case of incomplete observation (i.e., information only available at several limited stations) as future research topics. Agrawal et al. [2], as mentioned previously, said that "*it is often time consuming and expensive to track items through a process*" and raised the question of how to utilize the "cross-sectional" information, which is the essence of diagnosability of an MMP given that sensors are distributed only on the selected stations.

The authors developed a state space model [10,11] to characterize the dimensional stream of variation in an MMP. Based on the model, Ding et al. [14] presented the methodology for root cause diagnosis given the end-of-line measurements and the assumption that a single fixture fault occurs. The conditions of diagnosability, under the single fault assumption, are also given in terms of the distinguishability among fault patterns with the presence of noises.

However, it is often the case that two or more faults occur simultaneously in a process with multiple stations. The number of combinations of simultaneous fault patterns will grow exponentially when more stations are considered. The multiple fault patterns are rarely orthogonal. Nor do they have clear distinction between each other with the presence of noises. More challenges are encountered in evaluating the diagnosability of an MMP when multiple simultaneous faults are considered.

1.3 Proposed Method. This paper aims to fill this gap by presenting a diagnosability study of MMPs on the basis of a stream-of-variation model in a generic setting. The study of diagnosability is similar to that of observability in dynamic systems since our state space model is of the same format as that in dynamic systems. A diagnosability matrix is constructed in terms of process/product design, and can be used to indicate if the root cause can be uniquely identified based on the cross-sectional data. As for a partial diagnosable system, the diagnosability index (analogous to observability index) is defined to quantitatively characterize system capability in fault diagnosis.

The paper unfolds as follows. Section 2 presents a brief review of the state space model with some discussions. Section 3 studies the system diagnosability of a generic MMP. Examples are used in Section 4 to compare different sensor distribution schemes. Finally, in Section 5 we discuss the implication of this study and summarize the results.

2 State Space Model

In this paper, we primarily focus on dimensional quality control in discrete part manufacturing systems such as assembly and machining processes. The root cause to be isolated is mainly due to fixture malfunction, which is identified as a major dimensional variation contribution during new product launch [15]. For instance, a workpiece is positioned by a set of locators $\{P_1, P_2\}$ in *x-z* plane. The design nominal position of the part is shown in Fig. 2(*a*). When one of the fixture locators $\{P_1, P_2\}$ malfunctions, deviations could be associated with the corresponding locating point(s). Figure 2(*b*) shows that $\delta P_2(z)$ is the deviation in *z*-direction associated with locator P_2 when P_2 malfunctions. As a result, the workpiece is also deviated from its nominal position. The fixture variation is defined in terms of the variance of the positional deviation at a locating point, say $\mathrm{var}(\delta P_2(z))$, where $\mathrm{var}(\cdot)$ is the variance of a random variable. A fixture is considered "faulty" if its variation is greater than the assigned threshold. Such a large fixture variation occurs due to the fact that the locator may be worn, loose, bent, or broken. The task of root cause iden-

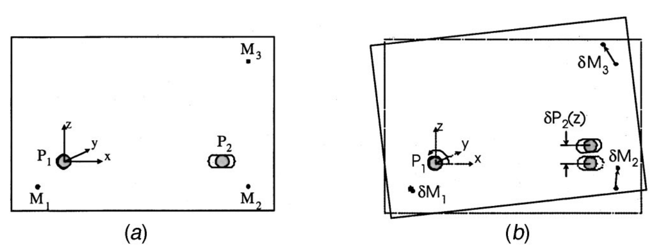

Fig. 2 An example of fixture fault manifestation

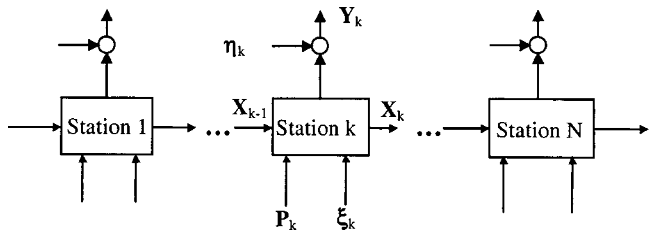

Fig. 3 Diagram of an assembly process with N stations

tification is to isolate the faulty fixture locator with undesired variation level. The diagnosability will be defined later in terms of fixture variation.

In order to deliver the intended dimensional accuracy of a product, dozens of fixtures are intensively used on each station throughout the production line. However, by adopting the product-inspection-oriented philosophy in quality assurance, fixtures used in production are not directly measured after being installed. The measurements that are taken on the finished product or intermediate products are values of product quality. The propagation of fixture variation contributed from each station and its impact on product quality are described by the stream-of-variation model.

Dimensional stream-of-variation model has been developed for multi-station assembly processes [10,11] and machining processes [16] by using state space representation. The stream of variation in an MMP is illustrated in Fig. 3 for an N-station process, where $\mathbf{X}_k \in R^{n\times 1}$ is the part accumulated deviation; $\mathbf{P}_k \in R^{m_k\times 1}$ is the fixture deviation contributed from station k; $\mathbf{Y}_k \in R^{q_k\times 1}$ is the measurement obtained on station k; superscripts n, m_k, and q_k are dimensions of the three vectors, respectively; $\boldsymbol{\xi}_k$, $\boldsymbol{\eta}_k$ are mutually independent noises.

The stream of variation in this MMP is characterized by the following equations

$$\mathbf{X}_k = \mathbf{A}_{k-1}\mathbf{X}_{k-1} + \mathbf{B}_k\mathbf{P}_k + \boldsymbol{\xi}_k, \quad k = 1,2,\ldots,N \tag{4}$$

$$\mathbf{Y}_k = \mathbf{C}_k\mathbf{X}_k + \boldsymbol{\eta}_k, \quad \{k\} \subset \{1,2,\ldots,N\} \tag{5}$$

where the first equation, known as the state equation, implies that part deviation on station k is influenced by two sources: the accumulated variation up to station $k-1$ and the variation contribution on station k; the second equation is the observation equation. Detailed expressions of system matrices $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}_k$ can be found in [10,11]. Regarding this representation, several remarks are articulated here.

Remark 2.1. Both $\mathbf{X}_k$ and $\mathbf{P}_k$ are random vectors. Since the variation-type faults are concerned in the dimensional control addressed in this paper, $\mathbf{X}_k$ and $\mathbf{P}_k$ are assumed to be zero-mean random vectors unless otherwise indicated.

Remark 2.2. System matrices $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}_k$ are determined by process/product design. $\mathbf{A}_k$, known as dynamic matrix, characterizes deviation change due to part transfer between station k and station $k+1$, namely $\mathbf{A}_k$ depends on the change of locating schemes in a production stream. If the fixture locating scheme is unchanged in the consecutive stations, e.g., several features are machined by using the same datum in a multi-station machining operation, then $\mathbf{A}_k$ is just an identity matrix $\mathbf{I}$ (corresponding to a simple translation) [16]. On the other hand, if a part is positioned by a new set of fixtures, the part will be reoriented on a new fixture set. As a result, $\mathbf{A}_k$ is no longer an identity matrix $\mathbf{I}$. More discussion on this topic is presented in [10,11].

Matrix $\mathbf{B}_k$ is the input matrix which determines how fixture deviation affects part deviation on station k, based on the geometry of a fixture locating layout. The rank of $\mathbf{B}_k$ equals to the number of degrees of freedom (d.o.f.) of the supported workpieces restrained by the fixture set.

Matrix $\mathbf{C}_k$ contains the information about sensor locations on a station. When sensors are installed on one or more stations in a production line, the index for the observation equation (Eq. 5) is actually a subset of $\{1,2,\ldots,N\}$, whereas the index for the state equation (Eq. 4) is the complete set. Similarly, the rank of $\mathbf{C}_k$ corresponds to the number of measured d.o.f. of a part or a subassembly on station k.

Matrices $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}_k$ in general are different for individual stations. Therefore, analogous to a dynamic system, the state space equations describe a discrete "time-varying" (actually station-varying) stochastic system. Therefore, the observability grammian $\mathbf{W}_0$ in Eq. (3), defined for a time invariant system, cannot be directly used here. The difference between the diagnosability of this station-indexed state space model and the observability of a time varying dynamic system will be discussed in detail in Section 3.

Remark 2.3. Also notice that an MMP has limited number of stations. An MMP is not only of finite horizon but also the number of states is small compared to a time series process. Processes such as an automotive body assembly process with 50 stations are already very complicated. Many other multi-station processes usually only constitute 10 stations or so. For such a small "run" station-indexed state space model, any techniques relying on convergence properties like the state observer in linear dynamic system will not work effectively since there is not enough "time" for the observer to converge to the true value.

In summary, the above remarks suggest that the stream-of-variation model integrates the process/product design and quality information. The model allows to apply the tools and concepts in system analysis to solve problems of manufacturing systems. The development of diagnosability analysis will use and expand mathematical tools of conventional system analysis.

3 Diagnosability Analysis

As it was stated earlier, there is similarity between concepts of diagnosability and observability. However, there are some essential differences between these two concepts for the state space model with station index. One difference is that the concept of observability relies on the complete measurements from the initial time t_0 to the current time [13], which accordingly requires the tracking of the same item on all stations in our station-indexed state space model. However, the diagnosability to be investigated here is based on measurements obtained from one (most likely the end-of-line) or several (but limited number) "cross-sections." Another difference is that the observability concept describes the state vector itself while the diagnosability concept focuses on the *variation* of state variables and input vectors. This further requires that the original state space model for variable deviations should be transformed into a state space model for the covariance matrix. It is assumed that accumulated part deviation $\mathbf{X}_{k-1}$, fixture deviation $\mathbf{P}_k$, and unmodeled higher order term $\boldsymbol{\xi}_k$ are independent. Thus, the state space model for the covariance matrix based on Eqs. (4) and (5) turns out to be

$$\boldsymbol{\Sigma}_k^{\mathrm{X}} = \mathbf{A}_{k-1}\boldsymbol{\Sigma}_{k-1}^{\mathrm{X}}\mathbf{A}_{k-1}^{\mathrm{T}} + \mathbf{B}_k\boldsymbol{\Sigma}_k^{\mathrm{P}}\mathrm{B}_k^{\mathrm{T}} + \boldsymbol{\Sigma}_k^{\xi}, \quad k = 1,2,\ldots,N \tag{6}$$

$$\boldsymbol{\Sigma}_k^{\mathrm{Y}} = \mathbf{C}_k\boldsymbol{\Sigma}_k^{\mathrm{X}}\mathbf{C}_k^{\mathrm{T}} + \boldsymbol{\Sigma}_k^{\eta}, \quad \{k\} \subset \{1,2,\ldots,N\} \tag{7}$$

where $\boldsymbol{\Sigma}_k^{(\cdot)}$ is the covariance matrix of a zero mean random vector on station k. Thus, in terms of the covariance matrix, the diagnosability of a multi-station system can be defined as follows.

Definition 1. The stream of variation in an MMP is called diagnosable if all fixture covariance matrices $\boldsymbol{\Sigma}_k^{\mathrm{P}}$, $k = 1,2,\ldots,N$, are uniquely determined given measurements $\boldsymbol{\Sigma}_k^{\mathrm{Y}}$ on selected stations, i.e., $\{k\} \subset \{1,2,\ldots,N\}$.

We first introduce the notation of state transition matrix $\boldsymbol{\Phi}_{(\cdot,\cdot)}$ from control theory as

$$\boldsymbol{\Phi}_{k,j} = \begin{cases} \mathbf{A}_{k-1}\mathbf{A}_{k-2}\cdots\mathbf{A}_j, & k \geq j+1 \\ \mathbf{I}, & k = j \end{cases}. \tag{8}$$

The state space model in Eqs. (4) and (5) can be converted into an input-output model

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_N \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1\mathbf{\Phi}_{1,0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix} \cdot \mathbf{X}_0 + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{bmatrix} \tag{9}$$

where $\mathbf{C}_k=\mathbf{0}$ if no sensor is installed at station k and $\boldsymbol{\varepsilon}_k=\Sigma_{i=1,k}\mathbf{C}_k\mathbf{\Phi}_{k,i}\boldsymbol{\xi}_i+\boldsymbol{\eta}_k$. Given the independence relationship assumed among $\mathbf{X}_{k-1}$, $\mathbf{P}_k$, $\boldsymbol{\xi}_k$, and $\boldsymbol{\eta}_k$, the product deviation $\mathbf{X}_0$, the fixture deviation $\mathbf{P}_k$, and the noise term $\boldsymbol{\varepsilon}_k$ are also independent. Further define

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix}, \quad \mathbf{\Gamma}_0 = \begin{bmatrix} \mathbf{C}_1\mathbf{\Phi}_{1,0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix} \tag{10}$$

we can have,

$$\mathbf{\Sigma}^{\mathrm{Y}} = \mathbf{\Gamma}\cdot\mathbf{\Sigma}^{\mathrm{P}}\cdot\mathbf{\Gamma}^T + \mathbf{\Gamma}_0\mathbf{\Sigma}_0^{\mathrm{X}}\mathbf{\Gamma}_0^T + \mathbf{\Sigma}^{\varepsilon} \tag{11}$$

where

$$\mathbf{\Sigma}^{\mathrm{P}} \equiv \mathrm{cov}([\mathbf{P}_1^T \quad \mathbf{P}_2^T \quad \cdots \quad \mathbf{P}_N^T]^T),$$
$$\mathbf{\Sigma}^{\mathrm{Y}} \equiv \mathrm{cov}([\mathbf{Y}_1^T \quad \mathbf{Y}_2^T \quad \cdots \quad \mathbf{Y}_N^T]^T),$$
$$\mathbf{\Sigma}^{\varepsilon} \equiv \mathrm{cov}([\boldsymbol{\varepsilon}_1^T \quad \boldsymbol{\varepsilon}_2^T \quad \cdots \quad \boldsymbol{\varepsilon}_N^T]^T),$$

and

$$\mathbf{\Sigma}_0^{\mathrm{X}} \equiv \mathrm{cov}(\mathbf{X}_0).$$

Then, the diagnosability in Definition 1 is equivalent to uniquely identifying $\mathbf{\Sigma}^{\mathrm{P}}$. In Eq. (11), $\mathbf{\Sigma}_0^{\mathrm{X}}$ is known based on measurements obtained at the end of the upstream process. For instance, in the case of assembly process, the upstream process is the sheet metal stamping process. If $\mathbf{\Sigma}_k^{\varepsilon}$ is not known and should be estimated through in-line measurements, more sensors need to be installed at appropriate locations within each station to reduce the influence of noise [17,18]. Those proposed techniques can be applied together with the diagnosability study presented in this paper to give a comprehensive solution. Given that those techniques are available, we assume in this study that $\mathbf{\Sigma}_k^{\varepsilon}$ is known or can be estimated from historical data. With this assumption, $\mathbf{\Sigma}^{\mathrm{LHS}}$ is defined as the summation of all measured or estimable quantities

$$\mathbf{\Sigma}^{\mathrm{LHS}} = \mathbf{\Sigma}^{\mathrm{Y}} - \mathbf{\Gamma}_0\mathbf{\Sigma}_0^{\mathrm{X}}\mathbf{\Gamma}_0^T - \mathbf{\Sigma}^{\varepsilon} \tag{12}$$

then, Eq. (11) is simplified as

$$\mathbf{\Sigma}^{\mathrm{LHS}} = \mathbf{\Gamma}\cdot\mathbf{\Sigma}^{\mathrm{P}}\cdot\mathbf{\Gamma}^T \tag{13}$$

where the right hand side is the summation of fixture variations from all stations, and is our focus in this study.

Since fixture deviations are independent among stations, $\mathbf{\Sigma}^{\mathrm{P}}$ is a diagonal matrix with the variances of fixture deviations as its diagonal elements. The diagonal elements are arranged in a variance vector $\boldsymbol{\sigma}^{\mathrm{P}}$ as

$$\begin{aligned}\boldsymbol{\sigma}^{\mathrm{P}} &= \mathrm{diag}(\mathbf{\Sigma}^{\mathrm{P}}) = [\mathrm{diag}(\mathbf{\Sigma}_1^{\mathrm{P}})^T \quad \cdots \quad \mathrm{diag}(\mathbf{\Sigma}_N^{\mathrm{P}})^T]^T \\ &= [\sigma_1^2(1) \quad \cdots \quad \sigma_{m_1}^2(1) \;\vdots\; \cdots \;\vdots\; \sigma_1^2(N) \quad \cdots \quad \sigma_{m_N}^2(N)]^T\end{aligned} \tag{14}$$

Then, Eq. (13) can be expressed as

$$\mathrm{vec}(\mathbf{\Sigma}^{\mathrm{LHS}}) = \pi(\mathbf{\Gamma})\cdot\boldsymbol{\sigma}^{\mathrm{P}} \tag{15}$$

where $\mathrm{vec}(\cdot)$ is the vector operator listed in Appendix I and $\pi(\cdot)$ is a transform defined as

$$\pi: \mathbf{\Gamma}^{q\times w} \to \pi(\mathbf{\Gamma})^{\frac{q(q+1)}{2}\times w} \tag{16}$$

$$\pi(\mathbf{\Gamma}) = \begin{bmatrix} \boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_q \\ \hline \boldsymbol{\gamma}_2\otimes\boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_2\otimes\boldsymbol{\gamma}_q \\ \hline \vdots \\ \hline \boldsymbol{\gamma}_q\otimes\boldsymbol{\gamma}_q \end{bmatrix} \tag{17}$$

where $\mathbf{\Gamma}\in R^{q\times w}$ and q, w are appropriate values corresponding to the dimensions of $\mathbf{\Gamma}$, $\boldsymbol{\gamma}_i$ is the i^{th} row vector of $\mathbf{\Gamma}$, and $\otimes$ represents the Hadamard product, explained in Appendix I as well. A brief derivation leading to Eq. (15) is included in Appendix II. The π-transform has the following properties that will be utilized later (the proofs are given in Appendix II):

Property 1. If the columns in $\mathbf{\Gamma}$ are independent, then the columns in $\pi(\mathbf{\Gamma})$ are also independent.

Property 2. Given any two matrices $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}_j$, if all columns in $\mathbf{\Gamma}_i$ are independent of those in $\mathbf{\Gamma}_j$, then all columns in $\pi(\mathbf{\Gamma}_i)$ will be independent of columns in $\pi(\mathbf{\Gamma}_j)$.

Then, we define the matrix

$$\mathbf{D}_N = \pi(\mathbf{\Gamma}) = \pi\left(\begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix}\right) \tag{18}$$

as the *diagnosability matrix*. The condition of the system diagnosability is revealed by the following theorem.

Theorem 1. Given an MMP characterized by Eqs. (4) and (5), the stream of variation is diagnosable if and only if $\mathbf{D}_N^T\mathbf{D}_N$ is of full rank or $\rho(\mathbf{D}_N)=\Sigma_{k=1}^N m_k$, where $\rho(\cdot)$ is the rank of a matrix and m_k is the number of potential fixture faults at station k.

The proof is straightforward and thus omitted. Because the π-transform conducts multiplication among rows, Eq. (18) can be further written as

$$\mathbf{D}_N = [\mathbf{\Pi}_1 \quad \cdots \quad \mathbf{\Pi}_k \quad \cdots \quad \mathbf{\Pi}_N] \tag{19}$$

where

$$\mathbf{\Pi}_k = \pi\left(\begin{bmatrix} \mathbf{0} \\ \hline \mathbf{C}_k\mathbf{B}_k \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,k}\mathbf{B}_k \end{bmatrix}\right) \tag{20}$$

$\mathbf{\Pi}_k$ is actually the diagnosability matrix of a single station k. It has a zero submatrix because the information obtained by sensors on stations $k-1,\ k-2,\ldots,1$ will not directly contribute to the diagnosis of faults on station k. The meaning of $\mathbf{\Pi}_k$ can be seen through the following. If only fixture fault ($\mathbf{P}_k$) on station k is concerned, we can write the input-output relationship as

$$\begin{bmatrix} \mathbf{Y}_k \\ \vdots \\ \mathbf{Y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{C}_k\mathbf{B}_k \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,k}\mathbf{B}_k \end{bmatrix} \cdot \mathbf{P}_k + \begin{bmatrix} \mathbf{C}_k\mathbf{\Phi}_{k,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix} \cdot \mathbf{X}_0 + \begin{bmatrix} \boldsymbol{\varepsilon}_k \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{bmatrix} \quad (21)$$

In order to make the sizes of vectors and matrices consistent when different stations are considered, i.e., to make the vector size the same as $[\mathbf{Y}_1^T \ \ldots \ \mathbf{Y}_N^T]^T$, we augment the vectors and matrices using a zero submatrix with appropriate dimensions as

$$\begin{bmatrix} \mathbf{0} \\ \hline \mathbf{Y}_k \\ \vdots \\ \mathbf{Y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{C}_k\mathbf{B}_k \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,k}\mathbf{B}_k \end{bmatrix} \cdot \mathbf{P}_k + \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{C}_k\mathbf{\Phi}_{k,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix} \cdot \mathbf{X}_0 + \begin{bmatrix} \mathbf{0} \\ \hline \boldsymbol{\varepsilon}_k \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{bmatrix} \quad (22)$$

Following the analysis procedure in Eqs. (9)–(15), $\mathbf{\Pi}_k$ will turn out to be the expression as defined in Eq. (20). It is obvious that $\mathbf{\Pi}_k$ characterizes the diagnosability of individual stations. This actually leads to a further partition of system diagnosability elaborated as follows.

Ding et al. [14] pointed out that the diagnosability of the entire system can be partitioned as two types: 1) within-station diagnosablility and 2) between-station diagnosability. The conditions of diagnosability given in [14] are for the single fault situation. Here, the conditions of diagnosability for multiple faults situation are stated.

Definition 2. The stream of variation is said to be within-station diagnosable (for instance, at station k) if all variances in $\boldsymbol{\sigma}_k^{\mathrm{P}} = \mathrm{diag}(\mathbf{\Sigma}_k^{\mathrm{P}}) = [\sigma_1^2(k) \ \cdots \ \sigma_{m_k}^2(k)]^T$ can be uniquely determined by measurements $\mathbf{\Sigma}_i^{\mathrm{Y}}$, $\{i\}\subset\{1,2,\ldots,N\}$ in the situation that fixture faults only occurred at that station.

Theorem 2. Given an MMP characterized by Eqs. (4) and (5), the stream of variation is within-station diagnosable at station k if and only if $\mathbf{\Pi}_k^T\mathbf{\Pi}_k$ is of full rank or $\rho(\mathbf{\Pi}_k)=m_k$.

Remark 3.1. $\mathbf{\Pi}_k$ is also called within-station diagnosability matrix. It is obvious that a station is not diagnosable if there is no sensor installed on or after this station.

Remark 3.2. The *observability matrix* [13] for a discrete time varying dynamic system on $[k_0,k_f]$ is

$$\mathbf{O}_{k_0,k_f} = \begin{bmatrix} \mathbf{C}_{k_0}\mathbf{\Phi}_{k_0,k_0} \\ \mathbf{C}_{k_0+1}\mathbf{\Phi}_{k_0+1,k_0} \\ \vdots \\ \mathbf{C}_{k_f-1}\mathbf{\Phi}_{k_f-1,k_0} \end{bmatrix} \quad (23)$$

Comparing the observability matrix $\mathbf{O}_{k_0,k_f}$ with diagnosability matrix $\mathbf{\Pi}_k$, we see several differences: (1) $\mathbf{\Pi}_k$ has matrix $\mathbf{B}_k$ at its each row element because diagnosability defined in this paper concerns input vector $\mathbf{P}_k$ rather than state vector $\mathbf{X}_k$. (2) Indices in row elements in $\mathbf{\Pi}_k$ may not be continuous because sensors are normally installed at selected stations in an MMP and each index corresponds to a physically different sensor or sensor set. On the other hand, indices in $\mathbf{O}_{k_0,k_f}$ are continuous from k_0 to k_f since sensors can continuously produce time-series data in dynamic system and all the indices actually correspond to the same sensor set. (3) The matrix $\mathbf{\Pi}_k$ is computed after applying the π-transform on the input-output matrix because diagnosability refers to the *variation of fixture deviation* rather than the fixture deviation itself. Based on the above comparison, it is concluded that within-station diagnosability matrix in an MMP is equivalent to the observability matrix defined in a dynamic system. The newly defined within-station diagnosability matrix captures the unique properties of manufacturing system, and thus it is more appropriate for manufacturing applications.

The between-station diagnosability refers to the capability to distinguish fixture faults occurring across different stations. If a fault occurred on station k, it may have distinct symptom from all other faults at the same and different stations. Then, the fault can be uniquely determined by using in-line measurements. However, it may have the same symptom with other faults at the same stations but the distinct symptom from faults at different stations. In such a situation, only a superposition of faults with the same symptom can be estimated. If all components included in the superposition come from the same station, we are still able to tell on which station the faulty fixture locates, although the exact faulty fixture is unclear.

Definition 3. The stream of variation is said to be between-station diagnosable on station k if a superposition of fixture faults on station k can be uniquely determined by measurements $\mathbf{\Sigma}_i^Y$, $\{i\}\subset\{1,2,\ldots,N\}$.

Theorem 3. Given an MMP characterized by Eqs. (4) and (5), the stream of variation is between-station diagnosable on station k if and only if the columns in $\mathbf{\Pi}_k$ are independent of all other columns in $\mathbf{D}_N$.

Proof. $\mathbf{D}_N$ is grouped into two parts: $\mathbf{\Pi}_k$ and $\mathbf{\Pi}^{(r)}$, where $\mathbf{\Pi}^{(r)}$ includes all the rest of the columns in $\mathbf{D}_N$ other than those in $\mathbf{\Pi}_k$. Accordingly, the variance vector $\boldsymbol{\sigma}^{\mathrm{P}}$ is also grouped as $[\boldsymbol{\sigma}_k^{\mathrm{T}}\boldsymbol{\sigma}^{(r)\mathrm{T}}]^{\mathrm{T}}$. By using the algorithm of singular value decomposition [19], we can write

$$\mathbf{\Pi}_k\boldsymbol{\sigma}_k = \sum_{i=1}^{\rho(\mathbf{\Pi}_k)} \lambda_i^{(k)}\mathbf{u}_i^{(k)}(\mathbf{v}_i^{(k)\mathrm{T}}\boldsymbol{\sigma}_k)$$

and

$$\mathbf{\Pi}^{(r)}\boldsymbol{\sigma}^{(r)} = \sum_{i=1}^{\rho(\mathbf{\Pi}^{(r)})} \lambda_i^{(r)}\mathbf{u}_i^{(r)}(\mathbf{v}_i^{(r)\mathrm{T}}\boldsymbol{\sigma}^{(r)}) \quad (24)$$

where $\mathbf{v}_i^{(k)\mathrm{T}}\boldsymbol{\sigma}_k$ is a linear combination of $\boldsymbol{\sigma}_k$. Since the columns in $\mathbf{\Pi}_k$ are independent of those in $\mathbf{\Pi}^{(r)}$, $\lambda_i^{(k)}\mathbf{u}_i^{(k)}$'s are also independent of $\lambda_i^{(r)}\mathbf{u}_i^{(r)}$'s, implying that the coefficient matrix comprising $\lambda_i^{(k)}\mathbf{u}_i^{(k)}$ and $\lambda_i^{(r)}\mathbf{u}_i^{(r)}$ is of full rank. Therefore, $\mathbf{v}_i^{(k)\mathrm{T}}\boldsymbol{\sigma}_k$, the linear combination of fixture faults at station k, i.e., the fault superposition on station k, is uniquely determined from the left hand vector in Eq. (24). Q.E.D.

If the stream of variation of an MMP is between-station diagnosable and also within-station diagnosable on each station, the entire system is then diagnosable. This partition provides a two-step diagnosis procedure in the MMP: first localize the fixture faults at a candidate station and second isolate the fixture fault right on that candidate station. This two-step procedure was proposed in [20] for a knowledge-based expert system. Our analytical approach is consistent with the heuristic reasoning procedure and provides more rigorous mathematics foundation.

When a system is not fully diagnosable, it can either be not diagnosable at all or partially diagnosable. Differentiation among these partially diagnosable system is also of our interest. Analogous to observability index [13] used in control theory, we define diagnosability index to quantitatively describe the percentage of potential faults that can be identified.

Definition 4. Within-station diagnosability index, denoted as μ_k, is defined as the ratio of the number of independent columns in $\mathbf{\Pi}_k$ over the number of potential fixture faults, that is,

$$\mu_k = \frac{\rho(\mathbf{\Pi}_k)}{m_k} \quad (25)$$

Index μ_k indicates how many fixture faults and/or their superposition on a station can be determined. It is a normalized quantity in $[0, 1]$. $\mu_k = 0$ means that the fixture fault on this station is completely undistinguishable, while $\mu_k = 1$ indicates a complete diagnosable station. Thus, Theorem 2 can be rephrased as that the stream of variation is within-station diagnosable on station k if and only if $\mu_k = 1$. Any μ_k between 0 and 1 is for a partially diagnosable station. In fact, the numerator of μ_k is corresponding to the observability index defined in control theory, where the observability index is not normalized. The difference between $\mathbf{\Pi}_k$ and observability matrix has already been addressed in *Remark 3.2.* In spite of the differences, these two indices have a common objective, which is to quantify the independent sensing information for a given system configuration. The independent sensing information refers to a sensor measurement which is distinct from and also not the superposition of other sensor measurements.

Definition 5. The process diagnosability index of an MMP, denoted as μ, is defined as

$$\mu = \frac{\rho(\mathbf{D}_N)}{\sum_{k=1}^{N} m_k} \tag{26}$$

Index μ quantitatively describes the percentage of independent equations with respect to the total number of potential fixture faults in the entire MMP. Same as μ_k, μ is also a normalized quantity in [0, 1], with $\mu = 1$ meaning that the process is completely diagnosable. It should be noted that the condition $\mu = 1$ is stronger than the condition $\{\mu_k = 1, k = 1, 2, \ldots, N\}$ because fixture faults could be undistinguishable among different stations, i.e., the between-station diagnosability not ensured.

4 Example

A three-station assembly process ($N=3$) is used to illustrate the proposed methodology. There are four parts marked as 1, 2, 3, and 4. Three stations are involved to finish assembly and measurements: (1) parts 1 and 2 are assembled at Station I (Fig. 4(*a*)), (2) subassembly "1+2" is welded with parts 3 and 4 at Station II (Fig. 4(*b*)), and (3) the assembly are measured at Station III (Fig. 4(*c*)). If I_k is used to denote the number of parts involved in the assembly on station k, then $I_1 = 2$, $I_2 = I_3 = 4$ for this example.

In this example, we only consider 2-D in-plane motion of rigid parts, i.e., DOF=3. Each part is restrained by a set of fixtures constituting a four-way pin/hole locating pair controlling motion in both x and z directions and a two-way pin/slot locating pair controlling motion only in z direction. A subassembly with several parts also needs a four-way pin and a two-way pin to completely control its d.o.f. For example, subassembly "1+2" is positioned by the fixture pair $\{P_1, P_4\}$ on part 1 and part 2, respectively, as shown in Fig. 4(*b*). A pinhole/slot is called "active" pinhole/slot if it is used on the current station to position an assembly, represented by a black circle/slot in Fig. 4; otherwise it is "inactive," represented by an empty circle/slot.

4.1 Sensor Distribution Scheme 1: End-of-Line Sensing. End-of-line sensing describes the scenario where all sensors are installed only at the end of production line. It is cost effective and widely implemented in industry for the purpose of product inspection.

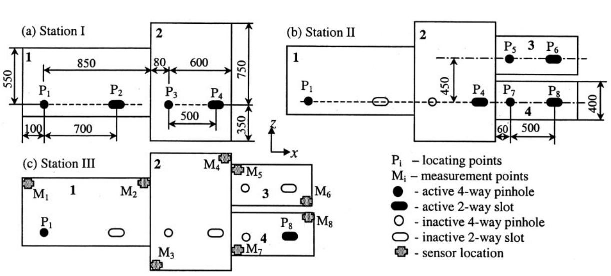

Fig. 4 Three-station assembly with end-of-line sensing

In such case, (after removing the zero submatrix) $\mathbf{\Pi}_k$ becomes

$$\mathbf{\Pi}_k = \pi(\mathbf{C}_3 \mathbf{\Phi}_{3,k} \mathbf{B}_k) \tag{27}$$

Even if $\rho(\mathbf{C}_3) = I_3 \cdot \text{DOF}$, i.e., a sufficient number of sensors can access all d.o.f. of workpieces in the product, $\mathbf{\Pi}_k$ could still be rank-deficient if $\mathbf{\Phi}_{3,k}$ is not of full rank. The only exception is for the last station $k=3$, where $\mathbf{\Pi}_3 = \pi(\mathbf{C}_3\mathbf{B}_3)$ since $\mathbf{\Phi}_{3,3} = \mathbf{I}$. The sufficient number of sensors makes all d.o.f. of components at Station III measurable, which renders $\rho(\mathbf{\Pi}_3) = m_3$. This suggests that the fixture fault is diagnosable at the last station. In general, the diagnosability of an MMP depends strongly on the transition matrix $\mathbf{\Phi}_{N,k}$, which is in turn determined by the dynamic matrix $\mathbf{A}_k$. As pointed out in *Remark 2.2*, $\mathbf{A}_k$ is primarily decided by the consistency in fixture layouts among stations. Difference of fixture locating layouts between stations results in reorientation so that some "memory" about part position and orientation on the previous station may be lost, causing $\mathbf{A}_k$ rank deficient. Thus, the system diagnosability is not guaranteed by employing the end-of-line sensing strategy.

The above argument can be verified by the given example, with sensors M_{1-8} installed at Station III as shown in Fig. 4. Here, $\rho(\mathbf{C}_3) = 4 \times 3 = 12$.

Step 1. Set up the state space model for this three-station process.

The state space representation of this process is shown as follows

$$\mathbf{X}_1 = \mathbf{B}_1 \mathbf{P}_1 + \boldsymbol{\xi}_1$$

$$\text{and} \quad \mathbf{X}_k = \mathbf{A}_{k-1}\mathbf{X}_{k-1} + \mathbf{B}_k\mathbf{P}_k + \boldsymbol{\xi}_k, \quad k = 2,3 \tag{28}$$

$$\mathbf{Y}_3 = \mathbf{C}_3 \mathbf{X}_3 + \boldsymbol{\eta}_3, \tag{29}$$

where the initial state $\mathbf{X}_0$ that is part deviation from the stamping process is assumed negligible. Numerical expressions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ of the assembly process in Fig. 4 are given as follows.

$$\mathbf{A}_1 = \left[\begin{array}{cccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0.0007 & 1 & 0 & -0.0007 & -0.3497 & \mathbf{0}^{6\times 6} \\
-1 & 0 & 0 & 1 & 0 & 0 & \\
0 & -0.3497 & 0 & 0 & 0.3497 & -325.17 & \\
0 & 0.0007 & 0 & 0 & -0.0007 & 0.6503 & \\
\hline
 & & & \mathbf{0}^{6\times 6} & & & \mathbf{I}^{6\times 6}
\end{array}\right]_{12\times 12}$$

$$\mathbf{A}_2 = \left[\begin{array}{ccc|c|ccc}
0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{0}^{3\times 6} & 0 & 0 & 0 \\
0 & 0.0005 & 1 & & 0 & -0.0005 & -0.2392 \\
\hline
-1 & 0 & 0 & & 0 & 0 & 0 \\
0 & -0.5550 & 0 & & 0 & -0.4450 & -222.49 \\
0 & 0.0005 & 0 & \mathbf{I}^{6\times 6} & 0 & -0.0005 & -0.2392 \\
-1 & -0.2153 & 0 & & 0 & 0.2153 & 107.655 \\
0 & -0.2392 & 0 & & 0 & -0.7608 & -380.38 \\
0 & 0.0005 & 0 & & 0 & -0.0005 & -0.2392 \\
\hline
-1 & 0 & 0 & & 1 & -0.0005 & 0 \\
0 & -0.2392 & 0 & \mathbf{0}^{3\times 6} & 0 & 0.2392 & -380.38 \\
0 & 0.0005 & 0 & & 0 & -0.0005 & 0.7608
\end{array}\right]_{12\times 12} \tag{30}$$

$$\mathbf{B}_1=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&-0.0014&0.0014&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&-0.002&0.002\\\hdashline\multicolumn{6}{c}{\mathbf{0}^{6\times6}}\end{bmatrix}_{12\times6}$$

$$\mathbf{B}_2=\left[\begin{array}{ccc:cccccc}1&0&0&&&&&&\\0&1&0&&&&&&\\0&-0.0007&0.0007&&&\mathbf{0}^{6\times6}&&&\\1&0&0&&&&&&\\0&0.3497&0.6503&&&&&&\\0&-0.0007&0.0007&&&&&&\\\hdashline&&&1&0&0&0&0&0\\&&&0&1&0&0&0&0\\&\mathbf{0}^{6\times3}&&0&-0.002&0.002&0&0&0\\&&&0&0&0&0&0&0\\&&&0&0&0&0&1&0\\&&&0&0&0&0&-0.002&0.002\end{array}\right]_{12\times9}\quad\mathbf{B}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&-0.0005&0.0005\\1&0&0\\0&0.5550&0.4450\\0&-0.0005&0.0005\\1&0.2153&-0.2153\\0&0.02392&0.7608\\0&-0.0005&0.0005\\1&0&0\\0&0.2392&0.7608\\0&-0.0005&0.0005\end{bmatrix}_{12\times3}\tag{31}$$

$$\mathbf{C}_3=\left[\begin{array}{c:c:c:c}\begin{matrix}1&0&-550\\0&1&-100\\1&0&-550\\0&1&850\end{matrix}&\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}\\\hdashline\mathbf{0}^{4\times3}&\begin{matrix}1&0&350\\0&1&-80\\1&0&750\\0&1&600\end{matrix}&\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}\\\hdashline\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}&\begin{matrix}1&0&-200\\0&1&-60\\1&0&200\\0&1&620\end{matrix}&\mathbf{0}^{4\times3}\\\hdashline\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}&\mathbf{0}^{4\times3}&\begin{matrix}1&0&200\\0&1&-60\\1&0&-200\\0&1&620\end{matrix}\end{array}\right]_{16\times12}\tag{32}$$

Step 2. Diagnosability Study. Within-station diagnosability matrix $\mathbf{\Pi}_k$ equal to $\pi(\mathbf{\Gamma}_k)$, where $\mathbf{\Gamma}_k=\mathbf{C}_3\mathbf{\Phi}_{3,k}\mathbf{B}_k$ for $k=1,2,3$. The $\mathbf{\Gamma}_k$'s are shown as follows,

$$
\mathbf{\Gamma}_1=\begin{bmatrix}
0 & 0.4011 & -0.7857 & 0 & 0 & 0.3846\\
0 & 0.0729 & -0.1429 & 0 & 0 & 0.0699\\
0 & 0.4011 & -0.7857 & 0 & 0 & 0.3846\\
0 & -0.6199 & 1.2143 & 0 & 0 & -0.5944\\
-1 & 0.2448 & 0 & 1 & -0.7 & 0.4552\\
0 & -0.4056 & 0 & 0 & 1.16 & -0.7544\\
-1 & -0.5245 & 0 & 1 & 1.5 & 0.9755\\
0 & 0.0699 & 0 & 0 & -0.2 & 0.1301\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{16\times 6}
$$

$$
\mathbf{\Gamma}_2=\begin{bmatrix}
0 & 0.1215 & -0.3846 & 0 & 0 & 0 & 0 & 0 & 0.2632\\
0 & 0.0221 & -0.0699 & 0 & 0 & 0 & 0 & 0 & 0.0478\\
0 & 0.1215 & -0.3846 & 0 & 0 & 0 & 0 & 0 & 0.2632\\
0 & -0.1877 & 0.5944 & 0 & 0 & 0 & 0 & 0 & -0.4067\\
0 & -0.0773 & 0.2448 & 0 & 0 & 0 & 0 & 0 & -0.1675\\
0 & -0.1877 & 0.5944 & 0 & 0 & 0 & 0 & 0 & -0.4067\\
0 & 0.1656 & -0.5245 & 0 & 0 & 0 & 0 & 0 & 0.3589\\
0 & -0.3379 & 1.0699 & 0 & 0 & 0 & 0 & 0 & -0.7321\\
-1 & -0.3110 & 0 & 1 & 0.4 & -0.4 & 0 & 0 & 0.3110\\
0 & -0.2679 & 0 & 0 & 1.12 & -0.12 & 0 & 0 & -0.7321\\
-1 & -0.1196 & 0 & 1 & -0.4 & 0.4 & 0 & 0 & 0.1196\\
0 & 0.0574 & 0 & 0 & -0.24 & 1.24 & 0 & 0 & -1.0574\\
-1 & 0.0957 & 0 & 0 & 0 & 0 & 1 & -0.4 & 0.3043\\
0 & -0.2679 & 0 & 0 & 0 & 0 & 0 & 1.12 & -0.8521\\
-1 & -0.0957 & 0 & 0 & 0 & 0 & 1 & 0.4 & -0.3043\\
0 & 0.0574 & 0 & 0 & 0 & 0 & 0 & -0.24 & 0.1826
\end{bmatrix}_{16\times 9}
$$

$$
\mathbf{\Gamma}_3=\begin{bmatrix}
1 & 0.2632 & -0.2632\\
0 & 1.0478 & -0.0478\\
1 & 0.2632 & -0.2632\\
0 & 0.5933 & 0.4067\\
1 & -0.1675 & 0.1675\\
0 & 0.5933 & 0.4067\\
1 & 0.3589 & -0.3589\\
0 & 0.2679 & 0.7321\\
1 & 0.3110 & -0.3110\\
0 & 0.2679 & 0.7321\\
1 & 0.1196 & -0.1196\\
0 & -0.0574 & 1.0574\\
1 & -0.0957 & 0.0957\\
0 & 0.2679 & 0.7321\\
1 & 0.0957 & -0.0957\\
0 & -0.0574 & 1.0574
\end{bmatrix}_{16\times 3} \tag{33}
$$

Table 1 Diagnosability indices

Parameters	Station I	Station II	Station III	Overall
$\rho(\cdot)$	4	9	3	16
m_k	6	9	3	18
μ	0.667	1	1	0.889

It is noticed that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are of rank 3 and rank 6, respectively, and $\mathbf{\Gamma}_3$ is of rank 3 (full rank). The reason that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are less than full rank is that state transition matrices $\mathbf{\Phi}_{(\cdot,\cdot)}$'s are singular for all transitions due to the inconsistency in fixture locating layouts among different stations. $\mathbf{\Gamma}_3$ is of full rank since the sensor is directly installed on Station III, and there is no part transition involved. Although $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are less than full rank, the corresponding within-station diagnosability matrices $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ could be of full rank since the π-transform defined in Appendix II can potentially increase the rank of a matrix by generating more rows and keeping the same number of columns. Actually, $\rho(\mathbf{\Pi}_1)=4$ and $\rho(\mathbf{\Pi}_2)=9$ so $\mathbf{\Pi}_2=\pi(\mathbf{\Gamma}_2)$ is of full rank. $\mathbf{\Pi}_3$ is certainly of full rank since $\mathbf{\Gamma}_3$ has full rank. Because of the high dimensions of $\mathbf{\Pi}_k$'s ($\mathbf{\Pi}_1\in R^{136\times6}$, $\mathbf{\Pi}_2\in R^{136\times9}$, $\mathbf{\Pi}_3\in R^{136\times3}$), their expressions are not listed.

One also notices that $\mathbf{\Gamma}_1$ can be divided into two blocks with dimension 8×6, and the second block of $\mathbf{\Gamma}_1$ is a zero matrix. The zero block is corresponding to parts 3 and 4, which have not been assembled at Station I. $\mathbf{\Gamma}_2$ and $\mathbf{\Gamma}_3$ do not contain such kind of zero block since all four workpieces appear on both Stations II and III. With the zero-block in $\mathbf{\Gamma}_1$, it is easy to verify that all columns in $\mathbf{\Gamma}_1$ are independent of those in $\mathbf{\Gamma}_2$ and $\mathbf{\Gamma}_3$. Moreover, with numerical computation, the columns in $\mathbf{\Gamma}_2$ are found to be independent to the columns in $\mathbf{\Gamma}_3$. According to *Property 2* of π-transform, the columns in $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$, or $\mathbf{\Pi}_3$ should be independent to those in the other two matrices. Therefore, the fixture fault in this assembly process is between-station diagnosable. The values of rank of $\mathbf{\Pi}_k$, m_k (i.e., # of faults), and diagnosability index are listed in Table 1. It is concluded that fixture fault is partially diagnosable at Station I but is diagnosable at Stations II and III. Thus, with the end-of-line sensing strategy, the entire system is not fully diagnosable. The overall system has 88.9% diagnosability, where $\mu_2=\mu_3=1$ and $\mu_1=0.667$ (66.7% diagnosability).

4.2 Sensor Distribution Scheme 2: "Saturated Sensing." When a sufficient number of sensors are installed at every station to measure all degrees of freedom of each part, i.e., $\rho(\mathbf{C}_k)=I_k\cdot$DOF, this distribution scheme is known as "saturated sensing." The k^{th} row block in $\mathbf{\Pi}_k$ is $\mathbf{C}_k\mathbf{B}_k$, which is the transformation from the fixture deviation to the deviation at the sensor location at station k. Since all d.o.f. of each workpiece are restrained and also measured, $\mathbf{C}_k\mathbf{B}_k$ is of full rank in this case. Thus, the corresponding columns in $\mathbf{\Pi}_k$ after the π-transform are independent due to *Property 1* of π-transform. As a result, $\rho(\mathbf{\Pi}_k)=m_k$, namely $\mathbf{\Pi}_k^T\mathbf{\Pi}_k$ is of full rank. Moreover, the row blocks in $\mathbf{\Pi}_k$ with index less than k are all zero blocks (refer to Eq. (20)), implying that $\mathbf{D}_N$ is in an echelon format and $\mathbf{\Pi}_j$ has different number of zero rows as $\mathbf{\Pi}_k$ has, for any $j\neq k$. According to this echelon structure, it is obvious that the columns in $\mathbf{\Pi}_k$ are independent of $\mathbf{\Pi}_j$ for any $j\neq k$. That means fixture fault is not only with-station diagnosable at each station but also between-station diagnosable, so it is equivalent to the condition that $\mathbf{D}_N$ is of full rank and $\mu=1$. This is consistent with our intuition that the complete diagnosability of MMPs can be achieved by installing sensors almost everywhere.

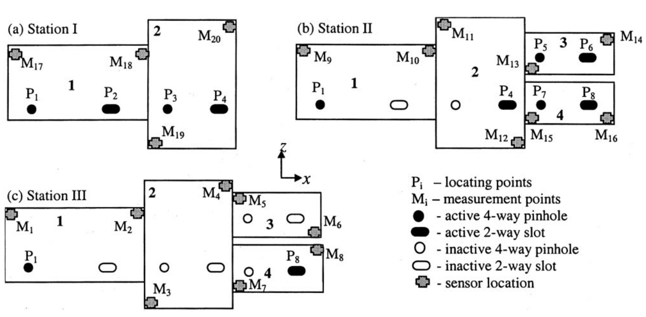

Fig. 5 Three-station assembly with "saturated sensing"

In this example, there are total 20 sensors installed at all three stations, with two sensors on each part, as shown in Fig. 5. The state equations are the same as those in Eq. (28). However, the observation equation consists of three equations for sensors installed at all three stations as

$$\mathbf{Y}_k=\mathbf{C}_k\mathbf{X}_k+\boldsymbol{\eta}_k,\quad k=1,2,3, \tag{34}$$

where $\mathbf{C}_3$ is the same as that in Eq. (32), and $\mathbf{C}_1$ and $\mathbf{C}_2$ are given as follows.

$$\mathbf{C}_1=\left[\begin{array}{ccc:ccc:c} 1 & 0 & -550 & & & & \\ 0 & 1 & -100 & & \mathbf{0}^{4\times3} & & \\ 1 & 0 & -550 & & & & \\ 0 & 1 & 850 & & & & \mathbf{0}^{8\times6} \\ \hdashline & & & 1 & 0 & -550 & \\ & \mathbf{0}^{4\times3} & & 0 & 1 & -100 & \\ & & & 1 & 0 & -550 & \\ & & & 0 & 1 & 850 & \end{array}\right]_{8\times12}$$

$$\mathbf{C}_2=\left[\begin{array}{ccc:ccc:ccc:ccc} 1 & 0 & -550 & & & & & & & & & \\ 0 & 1 & -100 & & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & \\ 1 & 0 & -550 & & & & & & & & & \\ 0 & 1 & 850 & & & & & & & & & \\ \hdashline & & & 1 & 0 & -750 & & & & & & \\ & \mathbf{0}^{4\times3} & & 0 & 1 & -80 & & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & \\ & & & 1 & 0 & 350 & & & & & & \\ & & & 0 & 1 & 600 & & & & & & \\ \hdashline & & & & & & 1 & 0 & 200 & & & \\ & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & & 0 & 1 & -60 & & \mathbf{0}^{4\times3} & \\ & & & & & & 1 & 0 & -200 & & & \\ & & & & & & 0 & 1 & 620 & & & \\ \hdashline & & & & & & & & & 1 & 0 & 200 \\ & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & & & \mathbf{0}^{4\times3} & & 0 & 1 & -60 \\ & & & & & & & & & 1 & 0 & 200 \\ & & & & & & & & & 0 & 1 & 620 \end{array}\right]_{16\times12} \tag{35}$$

The above generic conclusion concerning the diagnosability of "saturated sensing" scheme can be confirmed by numerical calculation. However, this scenario is too costly in reality and not necessary.

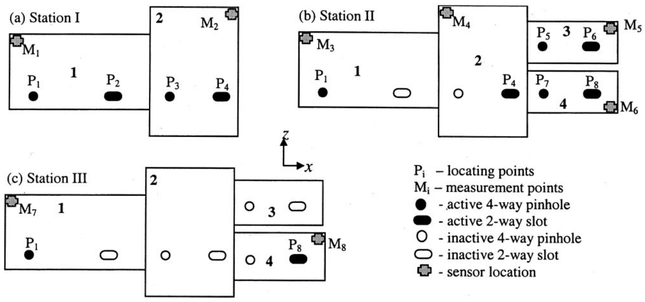

Fig. 6 Sensor distribution scheme 3

4.3 Sensor Distribution Scheme 3. We know that the sensor installation in Scheme 2 can achieve $\mu_k=1$ for all three stations. Using the sensor installation in Scheme 2 as the starting point, Scheme 3 can be obtained by trying out different combinations to minimize the number of sensors while maintain $\mu_k=1$ at each station. Since the number of sensors is limited at each station, the minimum solution usually can be quickly found for a single station. In this procedure, minimization is only done for a single station without considering sensors installed on other stations. Therefore, it is a local optimization scheme.

As for Scheme 3, the station-wise optimization is carried out forward from Station I to Station III. The results are shown in Fig. 6, with total 8 sensors installed at three stations. At Station I, two sensors ($M_{1,2}$) are placed on part 1 and part 2, respectively. At Station II, four sensors (M_{3-6}) are placed on different parts with one sensor on each individual part. Two more sensors ($M_{7,8}$) are placed on part 1 and part 4 at Station III, respectively. Though $\{\mu_k=1, k=1,2,3\}$ is not in general equivalent to $\mu=1$, μ is in fact 1 for this specific sensor distribution, after achieving $\mu_k=1$ for three stations.

The observation matrices are different from those in Section 4.2 since the different number of sensors are implemented. Matrices $\mathbf{C}_1'$, $\mathbf{C}_2'$, and $\mathbf{C}_3'$ in $\mathbf{Y}_k=\mathbf{C}_k'\mathbf{X}_k+\boldsymbol{\eta}_k$, $k=1,2,3$ are given as

$$\mathbf{C}_1'=\left[\begin{array}{ccc|ccc|c} 1 & 0 & -550 & & \mathbf{0}^{2\times3} & & \\ 0 & 1 & -100 & & & & \mathbf{0}^{4\times6} \\ \hline & \mathbf{0}^{2\times3} & & 1 & 0 & -750 & \\ & & & 0 & 1 & 600 & \end{array}\right]_{4\times12}$$

$$\mathbf{C}_2'=\left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & -550 & & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & \\ 0 & 1 & -100 & & & & & & & & & \\ \hline & \mathbf{0}^{2\times3} & & 1 & 0 & -750 & & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & \\ & & & 0 & 1 & -80 & & & & & & \\ \hline & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & & 1 & 0 & -200 & & \mathbf{0}^{2\times3} & \\ & & & & & & 0 & 1 & 620 & & & \\ \hline & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & & & \mathbf{0}^{2\times3} & & 1 & 0 & 200 \\ & & & & & & & & & 0 & 1 & 620 \end{array}\right]_{8\times12}$$

$$\mathbf{C}_3'=\left[\begin{array}{ccc|c|c|ccc} 1 & 0 & -550 & \mathbf{0}^{2\times3} & \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} & \\ 0 & 1 & -100 & & & & & \\ \hline & \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} & \mathbf{0}^{2\times3} & 1 & 0 & -200 \\ & & & & & 0 & 1 & 620 \end{array}\right]_{4\times12} \quad (36)$$

By using this sensor distribution scheme, the number of sensors implemented is 12 less than that in the "saturated sensing" scheme, rendering a 60% decrease.

Table 2 Comparison among four sensing schemes

	# of sensing station	# of sensor	μ
Scheme 1: end-of-line sensing	1	8	88.9%
Scheme 2: saturated sensing	3	20	100%
Scheme 3	3	8	100%
Scheme 4	3	5	100%

4.4 Sensor Distribution Scheme 4. Distribution Scheme 4 is obtained in a similar way as Scheme 3, however, with the station-wise optimization done backward from Station III to Station I. On each single station, the procedure is the same as that in Scheme 3. From the diagnosability study, it is known that sensors installed on the downstream station will contribute to diagnosis of fixture fault at the previous stations, whereas sensors on upstream stations do not help in diagnosing the downstream process. As a result, sensors distributed by using Scheme 4 should be no more than the number of sensors determined in Scheme 3.

Scheme 4 ends up with five sensors $\{M_{1-5}\}$ implemented in the process to achieve 100% within-station diagnosability, i.e., $\mu_k=1$, $k=1,2,3$. The five sensors are distributed at three stations with two at Station III, two at Station II, and one at Station I, shown in Fig. 7. When the overall system diagnosability is verified, μ is also 100% for this particular example.

The observation matrices $\mathbf{C}_1''$, $\mathbf{C}_2''$, and $\mathbf{C}_3''$ in $\mathbf{Y}_k=\mathbf{C}_k''\mathbf{X}_k+\boldsymbol{\eta}_k$, $k=1, 2, 3$ are different from those in Sections 4.2 and 4.3, and are also listed here.

$$\mathbf{C}_1''=\left[\begin{array}{ccc|c} 1 & 0 & -550 & \mathbf{0}^{2\times9} \\ 0 & 1 & -100 & \end{array}\right]_{2\times12}$$

$$\mathbf{C}_2''=\left[\begin{array}{c|ccc|ccc|c} \mathbf{0}^{2\times3} & 1 & 0 & -750 & & \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} \\ & 0 & 1 & -80 & & & & \\ \hline \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} & & 1 & 0 & -200 & \mathbf{0}^{2\times3} \\ & & & & 0 & 1 & 620 & \end{array}\right]_{4\times12}$$

$$\mathbf{C}_3''=\left[\begin{array}{ccc|c|c|ccc} 1 & 0 & -550 & \mathbf{0}^{2\times3} & \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} & \\ 0 & 1 & -100 & & & & & \\ \hline & \mathbf{0}^{2\times3} & & \mathbf{0}^{2\times3} & \mathbf{0}^{2\times3} & 1 & 0 & -200 \\ & & & & & 0 & 1 & 620 \end{array}\right]_{4\times12} \quad (37)$$

4.5 Comparison. Four schemes are compared in Table 2. Among these four given sensing schemes, Scheme 4 yields the least number of sensors while making $\mu=1$.

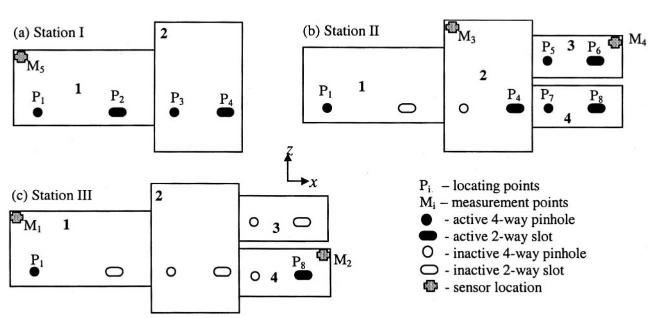

Fig. 7 Sensor distribution scheme 4

However, the limitations behind the station-wise optimization are: (1) The global optimality is not guaranteed since there may be further reduction in the number of sensing stations and sensors. (2) The two station-wise optimization algorithms, although yield systems with $\mu=1$, cannot in general guarantee a fully diagnosable system due to the fact that $\{\mu=1\}$ is a stronger condition than $\{\mu_k=1 \text{ for } k=1,2,\ldots,N\}$. In light of this need, more systematic study is desired to develop an optimal sensing strategy. The development of diagnosability analysis in this paper can be used as the mathematical basis for such research.

5 Conclusion

It is often costly and difficult to identify the root causes of the stream of variation in complex MMPs. A comprehensive diagnosability study based on the stream-of-variation theory is presented in this paper to address this issue.

The approach takes advantage of the state space system model that was developed in the authors' previous publications. In parallel to the concept of observability in control theory, diagnosability matrix and index are defined accordingly for MMPs. The diagnosability of the entire systems can be broken down at two levels—within-station diagnosability and between-station diagnosability. It turns out that the observability in control theory is corresponding to within-station diagnosability in manufacturing systems. This methodology can be used to evaluate different sensor distribution strategies. Through studies of the sensor distribution schemes of end-of-line sensing and ''saturated sensing,'' it is known that diagnosability is guaranteed but at a very high cost with a sufficient number of sensors installed at every station, while the end-of-line sensing, although economical, by no means guarantees system diagnosability. Other sensor distribution schemes that are discussed in this paper can reduce the number of sensors but cannot provide a global optimality and may sometimes result in $\mu<1$ by conducting station-wise optimization. In addition to this diagnosability study, more systematic study is needed to determine the optimal sensor distribution strategy in an MMP.

The presented methodology is fairly general for any MMP that can be modeled in a state space representation. The physical interpretation of the stream of variation relies on specific manufacturing systems. Unique properties of those processes should be taken into account so that the diagnosability study of the stream of variation can be carried over in different types of processes within the proposed framework.

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Appendix I: Hadamard Product and Vec Operator

The notations of Hadamard product and vec operator are detailed in [21]. Hadamard product, denoted by the symbol $\otimes$, simply performs the elementwise multiplication of two matrices. Given matrices $\mathbf{U}$ and $\mathbf{V}$ are each $m\times n$, then their Hadamard product is

$$\mathbf{U}\otimes\mathbf{V}=\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mn}\end{bmatrix}\otimes\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mn}\end{bmatrix} =\begin{bmatrix} u_{11}v_{11} & \cdots & u_{1n}v_{1n} \\ \vdots & \ddots & \vdots \\ u_{m1}v_{m1} & \cdots & u_{mn}v_{mn}\end{bmatrix} \tag{a1}$$

Some elementary properties that follow directly from the definition are listed as follows without proof, where matrices $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ are of the same size,

Item (a) $\mathbf{U}\otimes\mathbf{V}=\mathbf{V}\otimes\mathbf{U}$, item (b) $(\mathbf{U}\otimes\mathbf{V})\otimes\mathbf{W}=\mathbf{U}\otimes(\mathbf{V}\otimes\mathbf{W})$, item (c) $(\mathbf{U}+\mathbf{V})\otimes\mathbf{W}=\mathbf{U}\otimes\mathbf{W}+\mathbf{V}\otimes\mathbf{W}$, item (d) $(\mathbf{U}\otimes\mathbf{V})^{\mathrm{T}}=\mathbf{U}^{\mathrm{T}}\otimes\mathbf{V}^{\mathrm{T}}$, item (e) $\mathbf{w}^{\mathrm{T}}(\mathbf{u}\otimes\mathbf{v})=(\mathbf{w}\otimes\mathbf{v})^{\mathrm{T}}\mathbf{u}=(\mathbf{w}\otimes\mathbf{u})^{\mathrm{T}}\mathbf{v}$ where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are $m\times 1$ vectors.

Vec operator is used to transform a matrix to a vector that has the elements of the matrix as its elements. If a matrix $\mathbf{U}^{m\times n}$ has $\mathbf{u}_i$ as its i^{th} column vector, the vec($\mathbf{U}$) is the $mn\times 1$ vector given by

$$\text{vec}(\mathbf{U})=\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \vdots\\ \mathbf{u}_n\end{bmatrix} \tag{a2}$$

Since covariance matrix $\boldsymbol{\Sigma}$ is $n\times n$ square symmetric matrix, there are redundant elements in vec($\boldsymbol{\Sigma}$). Elimination of these redundant elements can reduce the dimension of vec($\boldsymbol{\Sigma}$) to $(n^2+n)/2\times 1$.

Appendix II: $\boldsymbol{\pi}$-Transform and Its Properties

Obtain Eq. (15) in Section 3. When $\boldsymbol{\Sigma}^{\mathrm{P}}$ is diagonal, we can write

$$\boldsymbol{\Sigma}^{\mathrm{P}}\cdot\boldsymbol{\Gamma}^{T}=[\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_1 \quad \cdots \quad \boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_q] \tag{a3}$$

Furthermore,

$$\boldsymbol{\Gamma}\cdot\boldsymbol{\Sigma}^{\mathrm{P}}\cdot\boldsymbol{\Gamma}=\begin{bmatrix} \boldsymbol{\gamma}_1(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_1) & \boldsymbol{\gamma}_1(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_2) & \cdots & \boldsymbol{\gamma}_1(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_q) \\ \boldsymbol{\gamma}_2(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_1) & \boldsymbol{\gamma}_2(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_2) & \cdots & \boldsymbol{\gamma}_2(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_q) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\gamma}_q(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_1) & \boldsymbol{\gamma}_q(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_2) & \cdots & \boldsymbol{\gamma}_q(\boldsymbol{\sigma}^{\mathrm{P}^T}\otimes\boldsymbol{\gamma}_q)\end{bmatrix} \tag{a4}$$

According to Property (e) of Hamadard product, Eq. (a4) turns out to be,

$$\boldsymbol{\Gamma}\cdot\boldsymbol{\Sigma}^{\mathrm{P}}\cdot\boldsymbol{\Gamma}^{T}=\begin{bmatrix} (\boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_1)\boldsymbol{\sigma}^{\mathrm{P}} & (\boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_2)\boldsymbol{\sigma}^{\mathrm{P}} & \cdots & (\boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_q)\boldsymbol{\sigma}^{\mathrm{P}} \\ (\boldsymbol{\gamma}_2\otimes\boldsymbol{\gamma}_1)\boldsymbol{\sigma}^{\mathrm{P}} & (\boldsymbol{\gamma}_2\otimes\boldsymbol{\gamma}_2)\boldsymbol{\sigma}^{\mathrm{P}} & \cdots & (\boldsymbol{\gamma}_2\otimes\boldsymbol{\gamma}_q)\boldsymbol{\sigma}^{\mathrm{P}} \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{\gamma}_q\otimes\boldsymbol{\gamma}_1)\boldsymbol{\sigma}^{\mathrm{P}} & (\boldsymbol{\gamma}_q\otimes\boldsymbol{\gamma}_2)\boldsymbol{\sigma}^{\mathrm{P}} & \cdots & (\boldsymbol{\gamma}_q\otimes\boldsymbol{\gamma}_q)\boldsymbol{\sigma}^{\mathrm{P}}\end{bmatrix} \tag{a5}$$

Since Hamadard product is inter-changeable (Property (a)), the matrix at right hand side in Eq. ($a5$) is symmetric, which is consistent with the symmetry of $\mathbf{\Gamma}\cdot\mathbf{\Sigma}^{\mathrm{P}}\cdot\mathbf{\Gamma}^{T}$. Rearranging Eq. ($a5$) in a vector format by using vec$(\cdot)$ operator will give

$$\mathrm{vec}(\mathbf{\Gamma}\cdot\mathbf{\Sigma}^{\mathrm{P}}\cdot\mathbf{\Gamma}^{T})=\pi(\mathbf{\Gamma})\cdot\boldsymbol{\sigma}^{\mathrm{P}} \qquad (a6)$$

Then,

$$\mathrm{vec}(\mathbf{\Sigma}^{\mathrm{LHS}})=\mathrm{vec}(\mathbf{\Gamma}\cdot\mathbf{\Sigma}^{\mathrm{P}}\cdot\mathbf{\Gamma}^{\mathrm{T}}) \qquad (a7)$$

which becomes Eq. (15) by substituting Eq. ($a6$) into it.

One should notice that π-transform can increase the rank of a matrix. For instance, given

$$\mathbf{\Gamma}_j=\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 0\\ 0 & -1 & 1\end{bmatrix} \quad \text{is of rank 2,} \qquad (a8)$$

after π transformation, we have

$$\pi(\mathbf{\Gamma}_j)=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}_{6\times 3} \quad \text{is of rank 3.} \qquad (a9)$$

This property is helpful in explaining why the variation can be diagnosable although $\mathbf{\Gamma}_j$ is singular, as we see in Section 4.1.

Proof of Property 1 in Section 3. Given that $\mathbf{\Gamma}$ is an $m\times n$ matrix, the first m rows in $\pi(\mathbf{\Gamma})$ is

$$\pi(\mathbf{\Gamma})_{1,m}=\begin{bmatrix}\boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_1\\ \vdots\\ \boldsymbol{\gamma}_1\otimes\boldsymbol{\gamma}_m\end{bmatrix} \qquad (a10)$$

where the subscript $(1, m)$ indicates that it is a submatrix containing row 1 to row m of the original matrix. Eq. ($a10$) can be written as

$$\pi(\mathbf{\Gamma})_{1,m}=\begin{bmatrix}| & | & & |\\ \gamma_{11}\cdot\boldsymbol{\gamma}^1 & \gamma_{12}\cdot\boldsymbol{\gamma}^2 & \cdots & \gamma_{1n}\cdot\boldsymbol{\gamma}^n\\ | & | & & |\end{bmatrix} \qquad (a11)$$

where $\boldsymbol{\gamma}^j$ is the j^{th} column vector of $\mathbf{\Gamma}$. Since the columns in $\mathbf{\Gamma}$ are independent, the columns in $\pi(\mathbf{\Gamma})_{1,m}$ should also be independent. Notice that

$$\pi(\mathbf{\Gamma})=\begin{bmatrix}\pi(\mathbf{\Gamma})_{1,m}\\ \pi(\mathbf{\Gamma})_{m+1,m(m+1)/2}\end{bmatrix} \qquad (a12)$$

Therefore, we can conclude the columns in $\pi(\mathbf{\Gamma})$ are independent.

Proof of Property 2 in Section 3. The proof of Property 2 will follow the same idea in the above proof of Property 1. Since the columns in $\mathbf{\Gamma}_i$ are independent of those in $\mathbf{\Gamma}_j$, it is easy to see that the columns in $\pi(\mathbf{\Gamma}_i)_{1,m}$ should be also independent of those in $\pi(\mathbf{\Gamma}_j)_{1,m}$. Again, partitioning $\pi(\mathbf{\Gamma}_i)$ and $\pi(\mathbf{\Gamma}_j)$ as

$$\pi(\mathbf{\Gamma}_i)=\begin{bmatrix}\pi(\mathbf{\Gamma}_i)_{1,m}\\ \pi(\mathbf{\Gamma}_i)_{m+1,m(m+1)/2}\end{bmatrix}$$

and

$$\pi(\mathbf{\Gamma}_j)=\begin{bmatrix}\pi(\mathbf{\Gamma}_j)_{1,m}\\ \pi(\mathbf{\Gamma}_j)_{m+1,m(m+1)/2}\end{bmatrix} \qquad (a13)$$

we can conclude that the columns in $\pi(\mathbf{\Gamma}_i)$ are independent of those in $\pi(\mathbf{\Gamma}_j)$.

Nomenclature

$\mathbf{A}_k$ = dynamic matrix
$\mathbf{B}_k$ = input matrix of station k
$\mathbf{C}_k$ = observation matrix of station k
$\mathbf{D}_N$ = the diagnosability matrix for the overall system
DOF = the degrees of freedom of each rigid workpiece, DOF=3 for a 2-D rigid body, DOF=6 for a 3-D rigid body
I_k = the number of parts involved in assembly at station k
N = the number of manufacturing stations
$\mathbf{P}_k$ = input vector, the fixture deviation vector of station k
$\mathbf{X}_k$ = state vector, the part deviation vector on station k
$\mathbf{Y}_k$ = observation vector on station k
d.o.f. = degrees of freedom
k = station index
n = dimension of $\mathbf{X}_k$, $\forall k$
m_k = dimension of $\mathbf{P}_k$
q_k = dimension of $\mathbf{Y}_k$
x, y, z = coordinate variables
vec$(\cdot)$ = vec operator
$\mathbf{\Sigma}$ = the covariance matrix
$\mathbf{\Phi}$ = state transition matrix
$\mathbf{\Pi}$ = within-station diagnosability matrix
$\mathbf{\Gamma}_i$ = equal to $\mathbf{C}_k\mathbf{\Phi}_{k,i}\mathbf{B}_i$
$\rho(\cdot)$ = the rank of a matrix
$\pi(\cdot)$ = π-transform
μ = diagnosability index
$\boldsymbol{\xi}, \boldsymbol{\eta}$ = noise vector

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