Multivariate Process Variability Monitoring Through Projection

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Inspired by the recently developed projection chart, such as the U^2 chart for monitoring a shift in the multivariate mean, this article proposes a multivariate projection chart for monitoring process variability. In engineering practice, people often build a linear process model to connect the multivariate quality measurements with a set of fixed assignable causes. The column space of the process model naturally provides a subspace for projection and subsequent monitoring and was indeed used as the projection subspace in the recently developed projection control charts for monitoring a shift the mean. For the purpose of monitoring variability, however, we will show that such a projection may not be advantageous. We propose an alternative projecting statistic, labeled as VS, to be used for constructing a multivariate variability monitoring chart. We show, through extensive numerical studies, that the VS chart entertains several advantages over other competing methods, such as its less restrictive requirements on the process model and generally improved detection performance.

Key Words: Control Chart; Generalized Variance; Multivariate Process Variability; Statistical Process Control.

PROCESS variability monitoring has been an important field in statistical quality control (Woodall and Montgomery (1999)). Several control charts have been developed specifically to detect the process-variability change. For example, R and S charts are used to detect the variance change in univariate measurements. For multivariate quality-characteristic measurements, the existing charts for monitoring variability are mainly based on the statistic of the generalized variance, calculated from the sample covariance matrix $\mathbf{S}$, such as $|\mathbf{S}|$ (Alt (1985)), $\log|\mathbf{S}|$ (Montgomery and Wadsworth (1972) and Alt and Smith (1988)). Recently, Reynolds and Cho (2006) proposed a multivariate exponentially weighted moving-average (MEWMA) chart for monitoring process variability. Besides these variability-monitoring charts, some other multivariate charts, such as Hotelling's T^2 chart, although designed for detecting mean shifts, can also signal variability changes because the T^2-chart has the sample covariance matrix $\mathbf{S}$ in its statistic. To use these control charts, measurements of quality characteristics are taken from the finished or intermediate product and are treated as random variables, and their distributions are compared with the corresponding distributions under normal conditions. If the measurements show that there are some quality characteristics "out of control" (e.g., deviation from the target or variability is too large), an alarm is generated to show that some faults happen in the process. Clearly, these charts are easy to use but do not take extra process information, such as the relationship between pro-

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cess faults and quality characteristics of products, into consideration.

There is some recent work in SPC to improve the monitoring performance through a subspace-projection method. Particularly, Runger (1996) proposed a projection chart, called a U^2 -chart, that appears to perform much better than a regular T^2-chart. This chart projects the quality measurement $\mathbf{y}$ into a predefined lower dimensional space and the projected vector is then monitored. The project directions can be identified by selecting a subset of quality variables that only certain assignable causes affect or by adopting a process model that links the model-fixed assignable causes with the quality variables. Quite a few such process models have been developed recently, for instance, the process-oriented basis-representation model (Barton and Gonzalez-Barreto (1996)), the physical models for assembly processes (Mantripragada and Whitney (1999), Jin and Shi (1999), Ding et al. (2000)), and the state-space models for machining processes (Zhou et al. (2003b), Djurdjanovic and Ni (2001), and Huang et al. (2000)). These models are in a common linear mixed-model form as

$$\mathbf{y} = \mathbf{A}\mathbf{f} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{y}$ is a vector consisting of product-quality measurements, $\mathbf{A}$ is a coefficient matrix determined by process/product design, $\mathbf{f}$ is a vector representing the model-fixed variation sources in the proces, and $\boldsymbol{\varepsilon}$ includes the measurement noise. In the implementation of the U^2-chart, the measurement $\mathbf{y}$ is projected onto the column space of matrix $\mathbf{A}$ and the projected results are monitored. It has been shown that the U^2-chart is quite sensitive in detecting the mean changes in $\mathbf{f}$. In a more recent paper, Runger et al. (2007) showed that the U^2-chart is actually equivalent to a T^2-chart on the weighted least squares estimation of $\mathbf{f}$ based on the model (1). Zhou et al. (2005) also proposed a directionally variant chart to take advantage of the known shift directions when assignable causes occur in the system. It is not surprising that these projection charts outperform the generic charts in detecting the changes in the root causes because extra information, the process model, is considered in the chart design.

The above-mentioned U^2-chart and the directionally variant chart are designed for mean-shift detection. Although the U^2-chart will also signal variance changes, further investigation is needed to differentiate whether the root cause is a mean shift or a variance change. However, in many practical situations, people are very interested in the variability change for variation-reduction purposes. Thus, it is highly desirable to develop a technique that is specifically tailored for variability-change detection. In other words, we want a control chart that is sensitive only to the variance change of $\mathbf{f}$, and it should outperform generic variability charts, such as the $|\mathbf{S}|$-chart, by taking the known process model into consideration.

To achieve this goal, we can follow exactly the procedure of the U^2-chart to, first, project $\mathbf{y}$ onto the column space of $\mathbf{A}$ and then monitor the variance of the projected values. In this paper, such a projection chart is called an $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart. However, the relationship between the covariance matrix of $\mathbf{y}$ and the covariance matrix of $\mathbf{f}$ is more complicated than the relationship between the mean of $\mathbf{y}$ and the mean of $\mathbf{f}$, which indicates a better way of projection should be established for the purpose of variability monitoring. In this paper, we propose a new projection chart based on the process model, called the VS (standing for variances summation) chart, to monitor the variance change of $\mathbf{f}$. The VS chart uses a different projection direction rather than the column space of $\mathbf{A}$ matrix, to take full advantage of the variational relation provided by the process model. In the implementation of the VS chart, we actually estimate the summation of the variances of the elements of $\mathbf{f}$ instead of needing to know each element of it. It is shown that the condition of the existence of VS is much more relaxed than that of $|\mathbf{S}_{\hat{\mathbf{f}}}|$, which requires all the columns of $\mathbf{A}$ to be independent of each other. In most cases, the performance of the VS chart is also superior to that of the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart.

The remainder of the article is organized as follows. In Section 2, the problem formulation and the development of the new projection chart are presented. Section 3 demonstrates the advantage of the proposed variability control chart using extensive numerical analyses. Finally, we conclude the paper in Section 4.

Development of $|\mathbf{S}_{\hat{\mathbf{f}}}|$ and VS Charts

All the above-mentioned models, which link the product quality measurements and process variation sources, are linear models. Linear models are widely used, not only because they are easy to deal with but also for the reason that a practical system usually operates closely around some nominal working point; hence, the system can be linearized without significant loss of accuracy. In this article, we also adopt the linear model shown in Equation (1) as our process model.

Without losing generality, we assume that $\mathbf{y} \in \mathbb{R}^q$, $\mathbf{f} \in \mathbb{R}^p$, $\boldsymbol{\varepsilon} \in \mathbb{R}^q$, and $\mathbf{A} \in \mathbb{R}^{q \times p}$. In addition, we assume that

(A1) $\mathbf{f}$ is normally distributed with zero mean and its p elements are independent, i.e., $\mathbf{f} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Sigma}_\mathbf{f})$, and $\boldsymbol{\Sigma}_\mathbf{f} = \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_p^2\}$; $\{\sigma_i^2\}_{i=1}^p$ are also called variance components.

(A2) $\boldsymbol{\varepsilon}$ is also normally distributed with zero mean and its variance–covariance matrix is a scalar matrix, i.e., $\boldsymbol{\varepsilon} \sim \mathrm{N}_q(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_q)$, where σ_ε^2 is the sensor noise variance and $\mathbf{I}_q$ is the identity matrix.

(A3) $\mathbf{f}$ and $\boldsymbol{\varepsilon}$ are independent.

(A4) The coefficient matrix $\mathbf{A}$ is a constant matrix, which is determined by the system, and $\mathbf{A}$ is assumed known.

These assumptions are not restrictive in practice and have been adopted by many other authors (e.g., Ceglarek and Shi (1996) and Apley and Shi (2001)). In (A1), we require each individual variation source (element in $\mathbf{f}$) be independent from each other, which is indeed the case in many manufacturing processes. For example, in an autobody-assembly process, the key process-variation sources are the fixture locator errors, and it is quite reasonable to assume the errors, such as breaking and loosening of different locators, are independent of each other. In (A2), we require the variability contribution of the noise on each measurement elements be the same. In many processes, $\boldsymbol{\varepsilon}$ is dominated by measurement noise. If the same measurement devices are used for each quality characteristic (e.g., the same coordinate-measurement machine is used to measure various dimensions on a car body), then (A2) is easily satisfied. Furthermore, because measurement noise is the major part of $\boldsymbol{\varepsilon}$, it is reasonable to assume $\mathbf{f}$ and $\boldsymbol{\varepsilon}$ are independent (A3). In (A4), it is assumed that the coefficient matrix $\mathbf{A}$ is known. This is not a restrictive assumption because $\mathbf{A}$ can often be either obtained from the physical analysis of the process or fitted empirically using historical data. In fact, as discussed in the introduction part, several techniques to obtain $\mathbf{A}$ for assembly processes, machining processes, and other discrete manufacturing processes have been developed recently.

With this process model and the assumptions, a straightforward application of the projection idea is to project the quality measurements $\mathbf{y}$ onto the column space of matrix $\mathbf{A}$. The projection procedure is the same as estimating the variance components (i.e., $\{\sigma_i^2\}_{i=1}^p$) using a least-squares estimation procedure (Ding et al. (2005)). The resulting chart is called the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart.

Chart $|\mathbf{S}_{\hat{\mathbf{f}}}|$

In the $|\mathbf{S}_{\hat{\mathbf{f}}}|$ chart, instead of monitoring the variability of $\mathbf{y}$, we monitor directly the variability of variation sources $\mathbf{f}$. However, because measurements of $\mathbf{f}$ are not available, we have to estimate $\mathbf{f}$ using the measurement $\mathbf{y}$ and the process model (1). In more detail, denote the linear minimum variance estimation of $\mathbf{f}$ as $\hat{\mathbf{f}}$, and then from Equation (1), we have

$$\hat{\mathbf{f}} = (\mathbf{A}^\mathrm{T}\mathbf{A})^{-1}\mathbf{A}^\mathrm{T}\mathbf{y}. \tag{2}$$

With a sample of size n, we have n measurement vectors $\{\mathbf{y}_i\}_{i=1}^n$ and hence n estimates of $\mathbf{f}$:

$$\hat{\mathbf{f}}_i = (\mathbf{A}^\mathrm{T}\mathbf{A})^{-1}\mathbf{A}^\mathrm{T}\mathbf{y}_i, \quad \text{for } i = 1, 2, \ldots, n. \tag{3}$$

Then the covariance matrix of $\hat{\mathbf{f}}$ is defined as

$$\mathbf{S}_{\hat{\mathbf{f}}} = \frac{1}{n-1}\sum_{i=1}^{n}(\hat{\mathbf{f}}_i - \bar{\mathbf{f}})(\hat{\mathbf{f}}_i - \bar{\mathbf{f}})^\mathrm{T}, \tag{4}$$

where $\bar{\mathbf{f}}$ is the average of $\{\hat{\mathbf{f}}_i\}_{i=1}^n$. Then $|\mathbf{S}_{\hat{\mathbf{f}}}|$ is the determinant of matrix $\mathbf{S}_{\hat{\mathbf{f}}}$.

Because the measurement $\mathbf{y}$ follows a normal distribution and the elements of $\hat{\mathbf{f}}$ are simply linear combinations of those of $\mathbf{y}$, according to Equation (2), $\hat{\mathbf{f}}$ is also normally distributed and thus $|\mathbf{S}_{\hat{\mathbf{f}}}|$ follows a similar distribution as $|\mathbf{S}_\mathbf{y}|$. Thus, we can setup the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart in a similar way to the $|\mathbf{S}_\mathbf{y}|$-chart.

Because $\hat{\mathbf{f}}$ is a p-variate normally distributed random vector, $|\mathbf{S}_{\hat{\mathbf{f}}}|$ has the same distribution as $[|\boldsymbol{\Sigma}_{\hat{\mathbf{f}}}|/(n-1)^p] Z_1 Z_1 \cdots Z_p$ (Djauhari (2005)), where $\{Z_i\}_{i=1}^p$ are independent random variables and $Z_i \sim \chi_{n-i}^2$ for $i = 1, \ldots, p$.

Alt (1985) proposed a control chart using the generalized variance $|\mathbf{S}|$ based on the fact that the distribution of $|\mathbf{S}|$ can be approximated by an asymptotic normal distribution. The center line of the chart is $\mathrm{E}(|\mathbf{S}|)$ and the control limits are $\mathrm{E}(|\mathbf{S}_{\hat{\mathbf{f}}}|) \pm 3\sqrt{\mathrm{Var}(|\mathbf{S}_{\hat{\mathbf{f}}}|)}$, where $\mathrm{E}(\cdot)$ and $\mathrm{Var}(\cdot)$ represent the mean and variance of a random variable. So, just like a generalized variance chart, we can set up the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart in a similar way, with the following control limits:

$$\mathrm{UCL} = \mathrm{E}(|\mathbf{S}_{\hat{\mathbf{f}}}|) + 3\sqrt{\mathrm{Var}(|\mathbf{S}_{\hat{\mathbf{f}}}|)}$$

$$\text{CL} = \text{E}(|\mathbf{S}_{\hat{\mathbf{f}}}|)$$
$$\text{UCL} = \text{E}(|\mathbf{S}_{\hat{\mathbf{f}}}|) - 3\sqrt{\text{Var}(|\mathbf{S}_{\hat{\mathbf{f}}}|)}. \quad (5)$$

The values of $\text{E}(|\mathbf{S}_{\hat{\mathbf{f}}}|)$ and $\text{Var}(|\mathbf{S}_{\hat{\mathbf{f}}}|)$ can be estimated in phase I analysis using a large number of regular samples. It is also worth mentioning that Djauhari (2005) recently proposed a method to determine the average of sample covariance matrices resulting in an unbiased estimate of the control limits for the generalized variance chart.

There are several interesting observations regarding the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart. The dimension of $\hat{\mathbf{f}}$, p, is usually smaller than that of $\mathbf{y}$, q, and thus the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart is usually more efficient than the $|\mathbf{S}_{\mathbf{y}}|$-chart in detecting the variation changes in $\hat{\mathbf{f}}$. This is also the key reason of why U^2 outperforms a generic T^2-chart. However, for multivariate variability monitoring, the $|\mathbf{S}_{\hat{\mathbf{f}}}|$ chart does have certain limitations. First, for $\mathbf{f}$ to be able to be estimated with Equation (2), matrix $\mathbf{A}$ has to have full column rank, i.e., the columns of matrix $\mathbf{A}$ have to be independent. Although this condition is satisfied by many engineering systems, it does not generally hold. In the case study, we will present a case in which this condition is violated. Second, by monitoring $|\mathbf{S}_{\hat{\mathbf{f}}}|$, we are actually detecting the changes in $|\mathbf{\Sigma}_{\hat{\mathbf{f}}}|$. However, $|\mathbf{\Sigma}_{\hat{\mathbf{f}}}|$ is different from $\mathbf{\Sigma}_{\mathbf{f}}$. In fact, according to Equations (1) and (2), we have

$$\mathbf{\Sigma}_{\mathbf{y}} = \mathbf{A}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{A}^{\mathrm{T}} + \mathbf{\Sigma}_{\varepsilon} \quad (6)$$
$$\begin{aligned}\mathbf{\Sigma}_{\hat{\mathbf{f}}} &= (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}_{\mathbf{y}}\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} \\ &= (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{A}^{\mathrm{T}} + \mathbf{\Sigma}_{\varepsilon})\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} \\ &= \mathbf{\Sigma}_{\mathbf{f}} + \sigma_{\varepsilon}^2(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}. \end{aligned} \quad (7)$$

Equation (7) indicates that $\mathbf{\Sigma}_{\hat{\mathbf{f}}}$ differs from $\mathbf{\Sigma}_{\mathbf{f}}$ by $\sigma_{\varepsilon}^2(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$. When the variance of measurement noise is small and matrix $\mathbf{A}$ is well conditioned (i.e., the values of the entries of $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ are small), then $\mathbf{\Sigma}_{\mathbf{f}}$ can be approximated by $\mathbf{\Sigma}_{\hat{\mathbf{f}}}$. In this case, the statistic $|\mathbf{S}_{\hat{\mathbf{f}}}|$ can be roughly viewed as detecting the changes in $\prod_{i=1}^{p}\sigma_i^2$. However, if the variance of the measurement noise is not small or dependency exists among the columns of matrix $\mathbf{A}$, which leads to large values of entries of $\sigma_{\varepsilon}^2(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$, then $\mathbf{\Sigma}_{\hat{\mathbf{f}}}$ will be influenced by $\sigma_{\varepsilon}^2(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ more heavily. As a consequence, the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart will become more sensitive to the changes in σ_{ε}^2 and less sensitive to the changes of $\{\sigma_i^2\}_{i=1}^{p}$.

VS Chart

The $|\mathbf{S}_{\hat{\mathbf{f}}}|$ chart uses the relationship of Equation (1) directly to conduct projection. It requires a somewhat strong condition on the coefficient matrix $\mathbf{A}$. When the matrix $\mathbf{A}$ does not have full column rank, the $|\mathbf{S}_{\hat{\mathbf{f}}}|$ chart cannot be employed. In contrast, the VS chart utilizes the variational relationship between $\mathbf{y}$ and $\mathbf{f}$ to conduct projection and thus can handle this situation better.

First, the variational relationship between $\mathbf{y}$ and $\mathbf{f}$ is obtained. Define $\sigma_{p+1}^2 = \sigma_{\varepsilon}^2$, then the covariance matrix of $\mathbf{y}$ can be written as

$$\mathbf{\Sigma}_{\mathbf{y}} = \mathbf{A}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{A}^{\mathrm{T}} + \mathbf{\Sigma}_{\boldsymbol{\varepsilon}} = \sum_{i=1}^{p+1}\sigma_i^2\mathbf{V}_i, \quad (8)$$

where $\mathbf{V}_i = \mathbf{a}_i\mathbf{a}_i^{\mathrm{T}}$ for $i = 1, \ldots, p$, $\mathbf{a}_i$ is the ith column of $\mathbf{A}$, and $\mathbf{V}_{p+1} = \mathbf{I}_q$ with $\mathbf{I}_q$ being the q-dimensional identity matrix.

Equation (8) can be further transformed into the following vector form:

$$\begin{aligned}\text{vec}(\mathbf{\Sigma}_{\mathbf{y}}) = \text{vec}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{f}}\mathbf{A}^{\mathrm{T}} + \mathbf{\Sigma}_{\boldsymbol{\varepsilon}}) &= \text{vec}\left(\sum_{i=1}^{p+1}\sigma_i^2\mathbf{V}_i\right) \\ &= \Pi(\mathbf{A})\cdot\boldsymbol{\sigma}, \end{aligned} \quad (9)$$

where vec(·) is the operator that stacks the columns of a matrix as a vector (Schott (2005)), $\boldsymbol{\sigma} = (\sigma_1^2 \mid \cdots \mid \sigma_p^2 \mid \sigma_{p+1}^2)^{\mathrm{T}}$, and $\Pi(\cdot)$ is a transform defined as

$$\begin{aligned}\Pi : \mathbb{R}^{q\times p} &\rightarrow \mathbb{R}^{q^2\times(p+1)}, \\ \mathbf{A} &\mapsto (\text{vec}(\mathbf{V}_1) \mid \cdots \mid \text{vec}(\mathbf{V}_p) \mid \text{vec}(\mathbf{I}_q)). \end{aligned} \quad (10)$$

From Equation (9), it is clear that, in order to estimate all the elements of $\boldsymbol{\sigma}$, i.e., the variances of each of the variation sources and the measurement noise, $\Pi(\mathbf{A})$ should have full column rank. The properties of $\Pi(\mathbf{A})$ have been studied intensively in recent diagnosability studies for variation source identification (Ding et al. (2002), Zhou et al. (2003a), and Ding et al. (2005)). It has been shown by Ding et al. (2005) that, if $\mathbf{A}$ is a tall matrix, i.e., $q \geq p + 1$, then a full rank matrix $\mathbf{A}$ will lead to a full rank matrix $\Pi(\mathbf{A})$, but not vice versa. In other words, there are situations when $\mathbf{A}$ is not full rank, but $\Pi(\mathbf{A})$ still satisfies full rank condition. For example, if

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

then rank($\mathbf{A}$) = 2, so $\mathbf{A}$ is a rank-deficient matrix.

However,

$$\Pi(\mathbf{A}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\text{rank}(\Pi(\mathbf{A})) = 4$, which means $\Pi(\mathbf{A})$ has full column rank.

The variational relationship in Equation (9) provides an alternative projection direction for monitoring the changes in the variance components: instead of projecting onto the column space of $\mathbf{A}$, which is suggested by model (1), we can project the elements of $\mathbf{\Sigma_y}$ onto the column space of $\Pi(\mathbf{A})$, which is suggested by the variational relationship (9). As follows, we show that the summation of the projection results is actually an estimation of the summation of elements of $\boldsymbol{\sigma}$, i.e., $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$, where $\mathbf{1}_{p+1}$ is a column vector of which the $p+1$ elements are all ones. Clearly, a simple monitoring method can be developed based on the estimates of $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$.

From the diagnosability study by Zhou et al. (2003a), we can estimate $\mathbf{q}^{\mathrm{T}}\boldsymbol{\sigma}$ based on Equation (9) if and only if $\mathbf{q}$ falls in the row space of $\Pi(\mathbf{A})$, where $\mathbf{q}$ is an arbitrary column vector. As a special case, then we know that $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$ can be estimated if $\mathbf{1}_{p+1}^{\mathrm{T}}$ falls in the row space of $\Pi(\mathbf{A})$. In other words, if

$$\text{rank}\left(\frac{\Pi(\mathbf{A})}{1 \quad 1 \quad \cdots \quad 1}\right) = \text{rank}(\Pi(\mathbf{A})) \tag{11}$$

holds, then $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$ is estimable. However, how to estimate $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$ is not discussed in the paper by Zhou et al. (2003a). In this paper, we propose an estimator based on the projection concept as

$$\tau \stackrel{\text{def}}{=} \mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\text{vec}(\mathbf{\Sigma_y}), \tag{12}$$

where $[\Pi(\mathbf{A})]^{+}$ is the Moore–Penrose inverse of $\Pi(\mathbf{A})$. It is shown in Appendix 1 that, if Equation (11) holds, then the scalar τ equals $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$. That is, the sum of elements of $\boldsymbol{\sigma}$ can always be calculated exactly using Equation (12).

In practice, $\mathbf{\Sigma_y}$ is unknown. However, the sample covariance matrix $\mathbf{S_y}$ is always available from the measurements and can be substituted into the place of $\mathbf{\Sigma_y}$ and results in a computable statistic that can be monitored as

$$\text{VS} \stackrel{\text{def}}{=} \mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\text{vec}(\mathbf{S_y}), \tag{13}$$

and we will call the control chart based on this statistic VS chart.

It can be shown that the statistic VS follows asymptotic normal distribution (please refer to Appendix 3 for the proof). With this in mind, the three-sigma control limits can be used when the sample size is large (e.g., a couple of hundred):

$$\begin{aligned} \text{UCL} &= \text{E(VS)} + 3\sqrt{\text{Var(VS)}} \\ \text{CL} &= \text{E(VS)} \\ \text{LCL} &= \max\left\{\text{E(VS)} - 3\sqrt{\text{Var(VS)}}, 0\right\}. \end{aligned} \tag{14}$$

The values of E(VS) and Var(VS) can be estimated in phase I analysis using a large number of regular samples. When sample size is small, VS deviates from the normal distribution. A simple simulation study shows that the VS distribution is flat compared with the normal distribution when the sample size is small and thus the false-alarm probability of the control chart in Equation (14) is higher than 0.0027. To reduce the false-alarm probability, wider control limits should be used. In practice, probability control limits can be obtained through empirical simulation when the sample size is small.

The statistic VS possesses some advantageous properties. First, if $\Pi(\mathbf{A})$ has full column rank, then the Moore–Penrose inverse $[\Pi(\mathbf{A})]^{+}$ is simply $[\Pi(\mathbf{A})^{\mathrm{T}}\Pi(\mathbf{A})]^{-1}\Pi(\mathbf{A})^{\mathrm{T}}$ and thus VS is the least-squares estimation of $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$ based on Equation (9). However, by using the Moore–Penrose inverse $[\Pi(\mathbf{A})]^{+}$, VS is always computable even if $[\Pi(\mathbf{A})^{\mathrm{T}}\Pi(\mathbf{A})]^{-1}$ does not exist. If Equation (11) cannot be satisfied, i.e.,

$$\text{rank}\left(\frac{\Pi(\mathbf{A})}{1 \quad 1 \quad \cdots \quad 1}\right) = \text{rank}(\Pi(\mathbf{A})) + 1,$$

then τ (and hence VS) is still computable, but τ becomes $\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \cdots + \alpha_{p+1}\sigma_{p+1}^2$, where $\{\alpha_i\}_{i=1,\ldots,p+1}$ are the elements of the vector

$$\boldsymbol{\alpha} \stackrel{\text{def}}{=} \mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}),$$

and they cannot be all ones. So the variable τ is a weighted sum of $\{\sigma_i^2\}_{i=1}^{p+1}$. The proof of this result can be found in Appendix 2. Second, it can also be shown that the condition of Equation (11) is more relaxed than the full column rank requirement of $\Pi(\mathbf{A})$. In other words, if $\Pi(\mathbf{A})$ has full column rank, then Equation (11) always holds, but the inverse is not true. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$\Pi(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$$

and $\mathrm{rank}(\Pi(\mathbf{A})) = 2$, which means $\Pi(\mathbf{A})$ does not have full column rank. However, the vector $\mathbf{1}_3^{\mathrm{T}}$ is clearly in the range space of $\Pi(\mathbf{A})^{\mathrm{T}}$.

In this section, we introduced the $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart and the VS chart for multivariate variability monitoring. The $|\mathbf{S}_{\hat{\mathbf{f}}}|$-chart can be viewed as the equivalent counterpart of the U^2-chart in variability monitoring. It requires a strong condition on the coefficient matrix $\mathbf{A}$. The VS chart makes the projection based on the variation relationship and needs a less strict assumption and can be implemented even if the system is not fully diagnosable. In the next section, the performance of these two charts will be compared with some existing multivariate variability-monitoring techniques through extensive numerical study.

Numerical Study

The process used in this case study is similar to the one used by Apley and Ding (2005). In that case, model (1) is used to describe the variation of the shape of the liftgate opening of a minivan. Vector $\mathbf{y}$ represents the measurement of the position of several points around the opening, $\mathbf{f}$ represents the variation patterns affecting the liftgate opening, and ε is the measurement noise. Six sensors are used, so $\mathbf{y}$ has six elements. There are a total of five variation patterns, so the dimension of $\mathbf{f}$ is five. Details about the system can be referred to Apley and Ding (2005). In the following case study, the system is slightly modified in order to better demonstrate the methods developed in this article. Three more sensors are added in addition to the original six such that we get a new matrix $\mathbf{A}$ of nine by five, as shown in the first column of Table 1. In this case, $\mathrm{rank}(\mathbf{A}) = 5$, which means $\mathbf{A}$ has full column rank and $\Pi(\mathbf{A})$ also has full column rank, hence $\hat{\mathbf{f}}$ and VS can be calculated using Equations (2) and (13), respectively. Furthermore, to demonstrate the performance of different charts when matrix $\mathbf{A}$ is not of full rank, two additional cases are constructed by modifying $\mathbf{A}$. The matrices for these two additional cases are listed in the second and third columns of Table 1. For case 2, $\mathrm{rank}(\mathbf{A}) = 4 < 5$ but $\Pi(\mathbf{A})$ is of full rank; for case 3, $\mathrm{rank}(\mathbf{A}) = 4 < 5$ and $\mathrm{rank}(\Pi(\mathbf{A})) = 5 < 6$, but

$$\mathrm{rank}\begin{pmatrix} \Pi(\mathbf{A}) \\ \hline 1 \;\; 1 \;\; \cdots \;\; 1 \end{pmatrix} = 5 = \mathrm{rank}(\Pi(\mathbf{A})),$$

so we can still set up the VS chart using Equation (13).

For each case, the average run lengths (ARLs) under out-of-control situations are compared among $|\mathbf{S}_{\mathbf{y}}|$, $|\mathbf{S}_{\hat{\mathbf{f}}}|$ (when applicable), and VS charts through numerical study. In the study, we select the standard deviation of measurement noise as 0.1; and when the minivan-assembly system operates in normal condition, the standard deviation of the variation sources (i.e., elements of $\mathbf{f}$) are also $\sigma_{\text{in-control}} = 0.1$. Based on this in-control situation, we can simulate the process to generate large amounts of measurement data when the process is normal, and then using the data, we can decide the control limits. Although the 3-σ limits are widely used for the $|\mathbf{S}_{\mathbf{y}}|$-chart, the performance of the chart using these control limits is very poor in these cases: a simple numerical study shows that the ARL_0 with sample size $n = 25$ is only 53.9 for the $|\mathbf{S}_{\mathbf{y}}|$-chart using 3-$\sigma$ control limits. To make fair comparisons, probability control limits are used in this study. To get the probability control limits, in phase I of this simulation, a large number of monitoring statistics for each control chart are produced when the process is in control, and the 0.135% and 99.865% percentile points of these statistics are used as the lower and upper control limits. The probability control limits for all three cases and three different sample sizes (25, 75, and 150) are listed in Table 2. These control limits will give an α-error of 0.0027 or ARL of 370 when the process is in control. In the ta-

TABLE 1. The Coefficient Matrices Used in the Three Simulation Cases

Matrix A of case 1	Matrix A of case 2	Matrix A of case 3
$\begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$	$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & -1 \end{pmatrix}^{\mathrm{T}}$	$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$

TABLE 2. Control Limits of Each Control Chart for Different Sample Sizes Case

Case		$\|\mathbf{S_y}\|$			$\|\mathbf{S_{\hat{f}}}\|$			VS		
		$n = 25$	$n = 75$	$n = 150$	$n = 25$	$n = 75$	$n = 150$	$n = 25$	$n = 75$	$n = 150$
1	UCL	2423.3	3077	2711	21.54	14.55	11.72	9.46	7.85	7.28
	LCL	6.6	143	325	0.34	1.53	2.44	3.43	4.43	4.87
2	UCL	4561.4	5768	5082	—	—	—	9.58	7.89	7.29
	LCL	12.4	268	608	—	—	—	3.55	4.48	4.89
3	UCL	5710	7221	6360	—	—	—	9.51	7.86	7.27
	LCL	15.5	336	762	—	—	—	3.57	4.49	4.90

ble, because $|\mathbf{S_{\hat{f}}}|$ is not available when the matrix **A** is not of full rank, we do not have $|\mathbf{S_{\hat{f}}}|$ chart for cases 2 and 3. Further, we notice that the control limits of $|\mathbf{S_y}|$ in cases 2 and 3 are significantly wider than that in case 1, which means that the structure of **A** influences significantly the distribution of the monitoring statistic of the $|\mathbf{S_y}|$-chart. By contrast, the probability control limits of the VS chart are fairly stable across different cases, suggesting that the monitoring statistics of the VS chart is insensitive to the model structure. This can be viewed as an advantage of the VS chart over the conventional $|\mathbf{S_y}|$-chart.

To compare the performance of these charts for different magnitudes of variability changes, we performed the simulations with $\sigma_i^2 = k \cdot \sigma^2_{\text{in-control}}$ for $i = 1, \ldots, 5$, and k running from 1.25 through 4. For each k and i, the ARL is obtained through Monte Carlo simulations and the overall ARL (illustrated in the following tables) is the average of the ARLs of different i, $i = 1, \ldots, 5$, but of the same k. Table 3 lists the ARL comparison for case 1 and Table 4 lists the results for both cases 2 and 3.

From these results, we observe the following. (1) The performance of the $|\mathbf{S_y}|$-chart is poor compared with the other two charts. It performs uniformly worse than the $|\mathbf{S_{\hat{f}}}|$-chart and is comparable with the VS chart only when both the variance change magnitude and the sample size are extremely large. However, when the sample size and/or the change magnitude is moderate, $|\mathbf{S_y}|$ performs significantly worse than the VS chart. This is not surprising because prior research has also argued that the projection method (here the $|\mathbf{S_{\hat{f}}}|$ and VS charts) can enhance the performance of monitoring by taking extra

TABLE 3. Comparison of ARL of Three Control Charts with Different Sample Sizes (Case 1)

k	$\|\mathbf{S_y}\|$			$\|\mathbf{S_{\hat{f}}}\|$			VS		
	$n = 25$	$n = 75$	$n = 150$	$n = 25$	$n = 75$	$n = 150$	$n = 25$	$n = 75$	$n = 150$
1	370.6	370.7	368.1	368.3	369.2	369.4	369.9	371.2	374.7
1.25	304.9	229.1	162.0	257.8	166.5	104.0	257.3	169.1	108.4
1.50	221.0	113.2	58.7	152.4	62.4	28.4	136.4	55.6	25.7
1.75	159.3	60.7	26.2	92.7	28.5	11.1	71.2	21.8	9.1
2	117.2	36.1	14.0	60.5	15.4	5.6	39.5	10.6	4.6
2.25	89.2	23.5	8.5	41.8	9.5	3.4	23.6	6.2	3.1
2.50	69.7	16.3	5.7	30.4	6.4	2.4	15.2	4.3	2.5
2.75	56.0	12.0	4.2	23.1	4.6	1.9	10.6	3.3	2.2
3	46.0	9.2	3.2	18.1	3.6	1.5	7.8	2.8	2.1
3.25	38.4	7.3	2.6	14.7	2.9	1.4	6.1	2.5	2.0
3.50	32.6	5.9	2.2	12.1	2.4	1.2	5.0	2.3	2.0
3.75	28.2	5.0	1.9	10.2	2.1	1.2	4.3	2.2	2.0
4	24.6	4.3	1.7	8.8	1.9	1.1	3.7	2.1	2.0

TABLE 4. Comparison of ARL of Two Control Charts with Different Sample Sizes (Cases 2 and 3)

	$\|\mathbf{S_y}\|$			VS		
k	$n = 25$	$n = 75$	$n = 150$	$n = 25$	$n = 75$	$n = 150$
			Case 2			
1	372.7	368.0	372.3	376.3	373.8	373.7
1.25	312.5	240.2	173.0	261.6	172.7	109.2
1.50	232.6	121.3	64.4	138.2	57.2	26.2
1.75	171.0	66.1	28.9	72.9	22.6	9.3
2	127.0	39.5	15.4	40.7	11.0	4.8
2.25	96.8	25.7	9.3	24.8	6.5	3.2
2.50	75.9	17.8	6.2	16.1	4.5	2.6
2.75	60.9	13.0	4.5	11.3	3.4	2.3
3	50.0	9.9	3.5	8.4	2.9	2.1
3.25	41.9	7.9	2.8	6.6	2.5	2.1
3.50	35.6	6.4	2.3	5.4	2.3	2.0
3.75	30.6	5.4	2.0	4.5	2.2	2.0
4	26.7	4.6	1.8	4.0	2.1	2.0
			Case 3			
1	372.3	371.4	369.8	369.9	372.9	371.0
1.25	314.0	244.6	181.8	255.2	167.1	105.4
1.50	238.1	132.5	75.8	134.1	54.5	25.1
1.75	179.3	77.1	37.0	69.8	21.5	9.0
2	136.6	48.3	20.7	39.0	10.5	4.6
2.25	106.7	32.4	12.8	23.6	6.3	3.1
2.50	85.5	22.9	8.6	15.4	4.3	2.5
2.75	69.9	17.0	6.1	10.8	3.3	2.2
3	57.9	13.0	4.7	8.0	2.8	2.1
3.25	49.0	10.3	3.7	6.3	2.5	2.0
3.50	41.9	8.4	3.0	5.2	2.3	2.0
3.75	36.4	7.0	2.6	4.4	2.2	2.0
4	31.9	5.9	2.2	3.8	2.1	2.0

model information (i.e., the subspace information) into consideration. (2) Comparing the $|\mathbf{S}_{\hat{\mathbf{f}}}|$ and VS charts, apparently the VS chart can be applied to a wider range of situations and requires a weaker condition on model matrix **A**. For example, the VS chart can be applied to both cases 2 and 3, while $|\mathbf{S}_{\hat{\mathbf{f}}}|$ cannot. Further, the performances of the VS chart in cases 2 and 3 are very similar to its performance in case 1. In other words, the deficiency in matrix **A** in cases 2 and 3 does not cause degradation in the performance of the chart. When the matrix **A** is of full rank, the performances of the $|\mathbf{S}_{\hat{\mathbf{f}}}|$ and the VS chart are quite similar, which is also expected. (3) The performance of the control charts are getting better when the sample size n increases; that is, the statistics are getting more and more sensitive to the shift of the process variability. (4) Another advantage of the VS chart is that the response of the VS chart to different faults is quite uniform. This cannot be readily observed in Tables 3 and 4 because the ARLs therein are averaged over different faults. Table 5 presents one example (case 1, $n = 75$) where the five individual ARLs with respect to the five individual faults, instead of their average, are displayed for different control charts. Clearly, the $|\mathbf{S_y}|$- and $|\mathbf{S}_{\hat{\mathbf{f}}}|$-charts are less sensitive to the occurrence of the third fault among these five faults, while the VS chart responds uniformly to these five faults, which can be

TABLE 5. Comparison of ARLs of the Control Charts for Each Fault (Case 1, $n = 75$)

	$\lvert\mathbf{S_y}\rvert$					$\lvert\mathbf{S_{\hat{f}}}\rvert$					VS				
k	#1	#2	#3	#4	#5	#1	#2	#3	#4	#5	#1	#2	#3	#4	#5
1	371.4	370.1	370.1	374.5	367.5	366.0	370.8	373.6	366.7	368.7	370.1	374.4	370.7	372.1	368.8
1.25	197.7	216.6	270.3	220.4	240.4	134.0	150.3	213.3	157.3	177.6	185.5	145.9	149.9	183.7	180.5
1.50	85.2	98.5	155.3	104.3	122.5	42.7	52.0	93.8	55.3	68.0	62.6	47.1	47.9	61.1	59.3
1.75	42.1	50.9	90.6	53.9	65.8	18.2	22.7	46.3	24.4	31.1	24.4	18.8	19.2	23.7	23.1
2	24.1	29.5	56.2	31.4	39.4	9.6	12.1	25.7	12.9	16.8	11.5	9.7	9.8	11.2	11.0
2.25	15.4	18.9	37.3	20.2	25.5	5.9	7.4	15.8	7.9	10.2	6.6	5.9	5.9	6.5	6.4
2.50	10.6	13.1	26.2	14.0	17.7	4.1	5.0	10.6	5.4	6.9	4.4	4.2	4.2	4.4	4.3
2.75	7.8	9.6	19.2	10.2	12.9	3.1	3.7	7.5	3.9	5.0	3.3	3.3	3.3	3.3	3.3
3	6.0	7.4	14.7	7.8	9.9	2.4	2.9	5.7	3.1	3.8	2.8	2.8	2.8	2.8	2.8
3.25	4.9	5.9	11.5	6.2	7.8	2.0	2.4	4.4	2.5	3.1	2.5	2.5	2.5	2.5	2.5
3.50	4.0	4.8	9.4	5.1	6.4	1.8	2.0	3.6	2.1	2.6	2.3	2.3	2.3	2.3	2.3
3.75	3.4	4.1	7.7	4.3	5.3	1.6	1.8	3.0	1.9	2.2	2.2	2.2	2.2	2.2	2.2
4	3.0	3.5	6.5	3.7	4.6	1.4	1.6	2.6	1.7	1.9	2.1	2.1	2.1	2.1	2.1

viewed as an advantage in general. This result can be explained by the fact that the monitoring statistic of the VS chart is actually an estimate of the summation of the variation components, σ_i^2, $i = 1, \ldots, 5$, and thus each variation component has equal weight in the statistic. These properties of VS make it the most desirable chart for variation monitoring when a process model is known.

Another important factor that influences the control-chart performance is the magnitude of the measurement noise. A signal-to-noise ratio (SNR), defined as the ratio of $\sigma^2_{\text{in-control}}$ to σ^2_ε (the ratio of in-control variance of process variables and that of measurement noise), is often used as an indicator of the magnitude of measurement noises. A similar definition is used in Ding et al. (2005). In the previous numerical study, because $\sigma^2_{\text{in-control}} = \sigma^2_\varepsilon = 0.1$, the SNR is 1. To study the influence of the measurement noise on the control-chart performance, we investigate values for SNR of 0.2, 1, and 5. In these studies, the sample size $n = 75$ is selected.

First, the probability control limits of the $|\mathbf{S_y}|$, $|\mathbf{S_{\hat{f}}}|$ (when applicable), and the charts for these SNRs are identified through simulation as above, and these control limits are shown in Table 6. From these control limits, it is apparent that the magnitude of measurement noise significantly influences the control limits of the $|\mathbf{S_y}|$- and $|\mathbf{S_{\hat{f}}}|$-charts, which indicates that the noise level significantly impacts the distribution of the monitoring statistic of these two charts. Once again, the control limits of VS are considerably stable across different SNR values. This suggests that the VS chart is more robust with respect to the disturbances of the measurement noise.

TABLE 6. Control Limits of Each Control Chart for Different SNR ($n = 75$)

Case		$\lvert\mathbf{S_y}\rvert$			$\lvert\mathbf{S_{\hat{f}}}\rvert$			VS		
		SNR = 0.2	SNR = 1	SNR = 5	SNR = 0.2	SNR = 1	SNR = 5	SNR = 0.2	SNR = 1	SNR = 5
1	UCL	6.68×10^7	3077	1.276	506.7	14.55	3.776	13.43	7.85	6.75
	LCL	3.12×10^6	143	0.059	53.1	1.53	0.397	7.06	4.43	3.90
2	UCL	1.20×10^8	5768	1.070	—	—	—	12.66	7.89	6.94
	LCL	5.56×10^6	268	0.050	—	—	—	7.82	4.48	3.81
3	UCL	1.58×10^8	7221	1.238	—	—	—	12.46	7.86	6.95
	LCL	7.35×10^6	336	0.057	—	—	—	7.95	4.49	3.79

TABLE 7. Comparison of ARL of Three Control Charts with Different SNR (Case 1, $n = 75$)

	$\|\mathbf{S_y}\|$			$\|\mathbf{S_{\hat{f}}}\|$			VS		
k	SNR = 0.2	SNR = 1	SNR = 5	SNR = 0.2	SNR = 1	SNR = 5	SNR = 0.2	SNR = 1	SNR = 5
1	366.6	370.7	370.6	372.2	369.2	368.5	371.2	371.2	373.1
1.25	303.2	229.1	186.1	269.3	166.5	122.5	273.0	169.1	136.2
1.50	216.7	113.2	76.1	161.3	62.4	37.5	154.0	55.6	37.7
1.75	151.7	60.7	37.2	97.8	28.5	15.8	83.0	21.8	14.0
2	108.0	36.1	21.1	62.4	15.4	8.4	46.8	10.6	6.9
2.25	79.1	23.5	13.5	41.8	9.5	5.2	28.0	6.2	4.3
2.50	59.6	16.3	9.3	29.4	6.4	3.6	17.9	4.3	3.2
2.75	45.9	12.0	6.9	21.4	4.6	2.7	12.1	3.3	2.7
3	36.2	9.2	5.4	16.2	3.6	2.2	8.7	2.8	2.4
3.25	29.0	7.3	4.3	12.5	2.9	1.9	6.6	2.5	2.2
3.50	23.8	5.9	3.6	10.0	2.4	1.6	5.3	2.3	2.1
3.75	19.8	5.0	3.1	8.1	2.1	1.5	4.4	2.2	2.1
4	16.6	4.3	2.7	6.8	1.9	1.4	3.7	2.1	2.0

Similar to the previous numerical study, the ARLs of the control charts under different conditions are identified through simulations. The results for case 1 are listed in Table 7 and those for cases 2 and 3 are listed in Table 8. The results are just as expected. First, when the SNR increases, all the control charts perform better because the statistics are less influenced by the measurement noise. Second, the $|\mathbf{S_{\hat{f}}}|$ and VS charts perform significantly better than the conventional $|\mathbf{S_y}|$-chart under all SNR levels. Furthermore, it seems that VS outperforms the $|\mathbf{S_{\hat{f}}}|$ chart when the noise level is high. This is consistent with the observation made based on the control limits: the VS chart is more robust to measurement noise perturbation.

Based on the above numerical study, we can see that, in most of the studied cases, the $|\mathbf{S_{\hat{f}}}|$ and VS charts outperform the $|\mathbf{S_y}|$-chart. The proposed projection chart using the VS statistic is preferred over the $|\mathbf{S_{\hat{f}}}|$-chart because it has broader applicability, performs more robustly to measurement noise perturbation, and responds more uniformly to different faults in the system.

Conclusion and Future Work

This research focuses on the development of a variance-monitoring control chart. In order to enhance the performance of the resulting control chart, the idea of projection methods is applied. Two projection charts, the $|\mathbf{S_{\hat{f}}}|$ and VS charts, were presented. Through a numerical study, both charts, as expected, demonstrate better performance than the conventional variance-monitoring chart. But the $|\mathbf{S_{\hat{f}}}|$-chart, using the same projection that is used in the U^2-chart (Runger (1996)), suffers a setback in its applicability, and sometimes in robustness as well, because it requires a stronger condition on the model matrix $\mathbf{A}$ to be computable. On the contrary, the VS chart, using a different projection method, which is more suitable for monitoring the variance components, entertains the advantages of being broadly applicable, robust to noise disturbance, and having uniform responses to different faulty inputs. For this reason, this VS control chart is recommended for practical use.

Future work can be done along the following lines. The distribution of VS is found to be asymptotic normally distributed. However, when the sample size is small, a systematic method for control-limits adjustment is needed to make sure the Type I error probability of the developed chart is at the specified level. Another interesting problem is the robustness of the proposed charts with respect to model uncertainty. Because these two charts heavily depend on the process model, the errors in the model will definitely have an impact on the performance of these two charts. The sensitivity of performance of these two charts to the model error is currently under investigation. The results will be reported in the near future.

TABLE 8. Comparison of ARL of Two Control Charts with Different SNR (Cases 2 and 3, n = 75)

	$\vert\mathbf{S_y}\vert$			VS		
k	SNR = 0.2	SNR = 1	SNR = 5	SNR = 0.2	SNR = 1	SNR = 5
Case 2						
1	370.1	368.0	371.4	372.5	373.8	368.9
1.25	295.4	240.2	216.7	233.3	172.7	155.6
1.50	196.8	121.3	101.2	105.0	57.2	47.7
1.75	127.2	66.1	52.6	49.1	22.6	18.4
2	85.0	39.5	30.7	25.6	11.0	9.1
2.25	58.8	25.7	19.7	14.7	6.5	5.4
2.50	42.5	17.8	13.6	9.4	4.5	3.8
2.75	31.7	13.0	10.0	6.5	3.4	3.1
3	24.4	9.9	7.7	4.9	2.9	2.6
3.25	19.3	7.9	6.1	4.0	2.5	2.4
3.50	15.6	6.4	5.0	3.3	2.3	2.2
3.75	12.9	5.4	4.2	2.9	2.2	2.1
4	10.8	4.6	3.7	2.7	2.1	2.1
Case 3						
1	371.4	371.4	372.0	370.8	372.9	368.9
1.25	296.9	244.6	223.3	216.4	167.1	155.9
1.50	198.6	132.5	114.0	91.1	54.5	48.1
1.75	130.2	77.1	63.9	40.7	21.5	18.5
2	88.4	48.3	39.5	20.8	10.5	9.2
2.25	62.1	32.4	26.2	12.0	6.3	5.5
2.50	45.4	22.9	18.5	7.7	4.3	3.9
2.75	34.2	17.0	13.7	5.5	3.3	3.1
3	26.5	13.0	10.5	4.2	2.8	2.6
3.25	21.1	10.3	8.3	3.5	2.5	2.4
3.50	17.1	8.4	6.8	3.0	2.3	2.2
3.75	14.1	7.0	5.7	2.7	2.2	2.1
4	11.8	5.9	4.8	2.5	2.1	2.1

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Appendix

1. If Equation (11) holds, then τ equals $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$.

If

$$\operatorname{rank}\begin{pmatrix}\Pi(\mathbf{A}) \\ \hline 1 \;\; 1 \;\; \cdots \;\; 1\end{pmatrix} = \operatorname{rank}(\Pi(\mathbf{A})),$$

then $\mathbf{1}_{p+1}$ is in the image space of $\Pi(\mathbf{A})^{\mathrm{T}}$, and there must exist a vector $\boldsymbol{\xi} \in \mathbb{R}^{q^2}$ such that $\mathbf{1}_{p+1} = \Pi(\mathbf{A})^{\mathrm{T}}\boldsymbol{\xi}$. Thus,

$$\begin{aligned} &\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) \\ &\quad = (\Pi(\mathbf{A})^{\mathrm{T}}\boldsymbol{\xi})^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) \\ &\quad = \boldsymbol{\xi}^{\mathrm{T}}\Pi(\mathbf{A})[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}). \end{aligned} \tag{15}$$

Because $[\Pi(\mathbf{A})]^{+}$ is the Moore–Penrose inverse of $\Pi(\mathbf{A})$, $\Pi(\mathbf{A}) \cdot [\Pi(\mathbf{A})]^{+} \cdot \Pi(\mathbf{A}) = \Pi(\mathbf{A})$, so Equation (15) can be written as

$$\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) = \boldsymbol{\xi}^{\mathrm{T}}\Pi(\mathbf{A}) = \mathbf{1}_{p+1}^{\mathrm{T}} \tag{16}$$

and

$$\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\mathrm{vec}(\mathbf{\Sigma_y}) = \mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}(\Pi(\mathbf{A})\boldsymbol{\sigma}) \\ = \mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}.$$

2. If Equation (11) does not hold, then τ cannot equal $\mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$.

To prove this statement, it is equivalent to proving that, if $\tau \equiv \mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$ for any $\boldsymbol{\sigma}$, condition Equation (11) is required.

If $\tau \equiv \mathbf{1}_{p+1}^{\mathrm{T}}\boldsymbol{\sigma}$, then $\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) = \mathbf{1}_{p+1}^{\mathrm{T}}$. Suppose $\mathrm{rank}(\Pi(\mathbf{A})) = \rho$, then $\Pi(\mathbf{A})$ can be decomposed as $\Pi(\mathbf{A}) = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}}$, where $\mathbf{U} \in \mathbb{R}^{q^2\times q^2}$ and $\mathbf{V} \in \mathbb{R}^{(p+1)\times(p+1)}$ are orthogonal matrices and $\mathbf{\Lambda} \in \mathbb{R}^{q^2\times(p+1)}$ is the singular-value matrix of which the first ρ diagonal elements are positive and other elements are zeros. Thus, $[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) = \mathbf{V}\mathbf{\Lambda}^{+}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}}$, so

$$\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}\Pi(\mathbf{A}) = \mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V}\mathbf{\Lambda}^{+}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}} = \mathbf{1}_{p+1}^{\mathrm{T}} \\ \mathbf{1}_{p+1}^{\mathrm{T}} = \mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V}\mathbf{\Lambda}^{+}\mathbf{\Lambda} = \mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V}.$$

Denote $\mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V} = (\,b_1 \quad b_2 \quad \cdots \quad b_{p+1}\,)$, then, because $\mathbf{\Lambda}^{+}\mathbf{\Lambda}$ is a square matrix of which the first ρ diagonal elements are ones and other elements are zeros,

$$\begin{aligned}\mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V} &= (\,b_1 \quad b_2 \quad \cdots \quad b_{p+1}\,) \\ &= \mathbf{1}_{p+1}^{\mathrm{T}}\mathbf{V}\mathbf{\Lambda}^{+}\mathbf{\Lambda} \\ &= (\,b_1 \quad b_2 \quad \cdots \quad b_{p+1}\,)\,\mathbf{\Lambda}^{+}\mathbf{\Lambda} \\ &= (\,b_1 \quad \cdots \quad b_\rho \quad 0 \quad \cdots \quad 0\,).\end{aligned}$$

That is, $b_{\rho+1} = b_{\rho+2} = \cdots = b_{p+1} = 0$. Hence, $\mathbf{1}_{p+1}^{\mathrm{T}}$ is in the space spanned by the first ρ rows of $\mathbf{V}^{\mathrm{T}}$, which is the same as the row space of $\Pi(\mathbf{A})$.

3. Distribution of VS

Define $\mathbf{1}_{p+1}^{\mathrm{T}}[\Pi(\mathbf{A})]^{+}$ as $\mathrm{vec}(\mathbf{H})^{\mathrm{T}}$, where $\mathbf{H}$ is a q by q is a square matrix. It can be shown that $\mathbf{H}$ is a symmetric matrix.

The proof is as follows:

$$\begin{aligned}\mathrm{vec}(\mathbf{H}^{\mathrm{T}}) &= \mathbf{K}\cdot\mathrm{vec}(\mathbf{H}) = \mathbf{K}[\Pi(\mathbf{A})^{\mathrm{T}}]^{+}\mathbf{1}_{p+1} \\ &= [\Pi(\mathbf{A})^{\mathrm{T}}\mathbf{K}^{-1}]^{+}\mathbf{1}_{p+1} \\ &= [\Pi(\mathbf{A})^{\mathrm{T}}]^{+}\mathbf{1}_{p+1} = \mathrm{vec}(\mathbf{H}),\end{aligned}$$

where $\mathbf{K} \in \mathbb{R}^{q^2\times q^2}$ is the commutation matrix. Thus, $\mathrm{VS} = \mathrm{vec}(\mathbf{H})^{\mathrm{T}}\mathrm{vec}(\mathbf{S_y}) = \mathrm{tr}(\mathbf{H}^{\mathrm{T}}\mathbf{S_y})$. When $\mathbf{H}$ is symmetric and $\mathbf{S_y}$ follows the Wishart distribution, Fujikoshi (1970) showed that the distribution of $\mathrm{tr}(\mathbf{H}^{\mathrm{T}}\mathbf{S_y})$ is asymptotically normal, so the statistic VS is asymptotically normally distributed.

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