# Optimal Sensor Distribution for Variation Diagnosis in Multistation Assembly Processes

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***Abstract*—This paper presents a methodology for optimal allocation of sensors in a multistation assembly process for the purpose of diagnosing in a timely manner variation sources that are responsible for product quality defects. A sensor system distributed in such a way can help manufacturers improve product quality while, at the same time, reducing process downtime. Traditional approaches in sensor optimization fall into two categories: multistation sensor allocation for the purpose of product inspection (rather than diagnosis); and allocation of sensors for the purpose of variation diagnosis but at a single measurement station. In our approach, sensing information from different measurement stations is integrated into a state-space model and the effectiveness of a distributed sensor system is quantified by a diagnosability index. This index is further studied in terms of variation transmissibility between stations as well as variation detectability at individual stations. Based on an understanding of the mechanism of variation propagation, we develop a backward-propagation strategy to determine the locations of measurement stations and the minimum number of sensors needed to achieve full diagnosability. An assembly example illustrates the methodology.**

***Index Terms*—Diagnosability, diagnosis of variation sources, multistation assembly process, sensor distribution.**

## NOMENCLATURE

| | |
|---|---|
| $\mathbf{A}_k$ | Dynamic matrix. |
| $\mathbf{B}_k$ | Input matrix of station $k$. |
| $\mathbf{C}_k$ | Observation matrix of station $k$. |
| $\mathbf{D}_N$ | Diagnosability matrix. |
| DOF | Degrees of freedom of each rigid workpiece, $\text{DOF} = 3$ for a 2-D rigid body, $\text{DOF} = 6$ for a 3-D rigid body. |
| $I_k$ | Number of parts involved in the assembly at station $k$. |
| $N$ | Number of manufacturing stations. |
| $\mathbf{P}_k$ | Input vector, the fixture deviation vector of station $k$. |
| $\mathbf{X}_k$ | State vector, the part deviation vector on station $k$. |
| $\mathbf{Y}_k$ | Observation vector on station $k$. |
| d.o.f. | Degrees of freedom. |
| $k$ | Station index. |
| $n$ | Dimension of $\mathbf{X}_k$ $\forall k$. |
| $n_p$ | Number of parts in an assembly. |
| $m_k$ | Dimension of $\mathbf{P}_k$. |
| $q_k$ | Dimension of $\mathbf{Y}_k$. |
| $s_j$ | Number of sensors on part $j$. |
| $x, y, z$ | Coordinate variables (translation). |
| $\text{vec}(\cdot)$ | Vector operator. |
| $\mathbf{\Sigma}$ | Covariance matrix. |
| $\mathbf{\Phi}$ | State transition matrix. |
| $\mathbf{\Gamma}$ | Input–output model matrix. |
| $\mathbf{\Gamma}_i$ | Equals $\mathbf{C}_k\mathbf{\Phi}_{k,i}\mathbf{B}_i$. |
| $\alpha, \beta, \omega$ | Coordinate variables (rotation). |
| $\rho(\cdot)$ | Rank of a matrix. |
| $\pi(\cdot)$ | $\pi$-transform. |
| $\mu$ | Diagnosability index. |
| $\lambda_{i\|k}$ | Transmissibility ratio. |
| $\tau$ | Detectability power. |
| $\boldsymbol{\xi}, \boldsymbol{\eta}$ | Noise vector. |

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## I. INTRODUCTION

RECENT innovations in sensor technology have enabled manufacturers to distribute sensors in multistation manufacturing processes. For example, optical coordinate measuring machines (OCMM) are built into automotive assembly lines and in-process CMMs are used in transfer-line machining processes. A distributed sensor system offers the enhanced capability of diagnosing in a timely manner process variation sources that cause product quality defects. A distributed sensor system enables manufacturers to improve product quality and reduce production downtime. However, effective use of sensing data in diagnosing variation sources depends to a great extent on the optimal design of the distributed sensor system. A poorly designed sensor system is likely to generate an extensive amount of irrelevant or even conflicting information and as such may not be able to provide the desired diagnosability in identifying variation sources.

The effectiveness of a sensor system is characterized by the diagnosability it offers, which is its capability to identify major variation sources. The efficiency of the system can be benchmarked by the sensing cost in achieving certain levels of diagnosability. The sensing cost considered in this paper is represented by the number of sensors and sensing stations that house them. The optimal design of a sensor system in terms of its effectiveness and efficiency is to realize the desired diagnosability at minimum cost.

TABLE I
COMPARISON OF SENSING DISTRIBUTION METHODOLOGIES

| **Problem Domain** | | | **Methodologies** |
|---|---|---|---|
| Quality Inspection -Oriented | Multi-stage Product Inspection | | Lindsay and Bishop [1], Britney [2], Eppen and Hurst [3], Yum and McDowell [4], Garcia-Diaz *et al.* [5], Yao and Zheng [7,8], Viswanadham *et al.* [9], Chen and Thornton [10], |
| Variation Diagnosis-Oriented | Single Station | | Udwadia [11], Fadale *et al.* [12], Khan *et al.* [13], Wang and Nagaerkar [14]. |
| | Multiple Stations | End-of-Line Sensing | Khan *et al.* [16]. |
| | | Distributed Sensing | Based on single fixture model and combinatorial optimization: Khan and Ceglarek [17]. Based on multi-station variation model and mechanism of variation propagation: ***To be presented in this paper*** |

In a multistation production line, the *design of a sensor system* involves the determination of (Q1) where to build sensing stations, (Q2) the minimal number of sensors required at each individual sensing station, and (Q3) the location of sensors within each individual sensing station. The design of such a sensor system with multiple sensing stations can be referred to as the problem of *sensor distribution* or *distributed sensing*. Please note that the "location of a sensor" refers to the location of a product feature that a sensor measures instead of the place where a sensor is physically installed. In other words, the number and location of sensors refers to the number and location of measurement features on a product/part. By "install a sensor on station $k$," we mean to select a product/part feature to be measured at station $k$. In this sense, the problem of designing a sensor system is selecting product features to measure on different stations in a multistation process.

Relevant research in this area falls into two major categories: sensor allocation for the purpose of multistage product inspection and, single-station sensing optimization. Optimal allocation of inspection efforts has been studied for serial and nonserial production lines with either perfect or imperfect inspection capability [1]–[5]. This type of research conducted prior to 1990 was summarized in a survey [6]. More recent development in this area is presented in [7], [8]. The objective of multistage product inspection is to minimize overall cost, including fixed inspection, variable inspection, scrap/repair and warranty costs. The problem is often formulated as a dynamic programming problem [6]. Other optimization methods used include nonlinear programming [4], genetic algorithms [9], and simulated annealing [9], [10].

In current industrial practices, product functional nonconformities are often the result of unsatisfactory quality. If the warranty cost is higher than other cost components, it can be more effective to minimize the sensing cost with quality requirements as constraints. A critical aspect of this new problem formulation is considering the mechanism and effectiveness of variation diagnosis, which leads to further diagnosis-oriented research in sensor optimization.

Diagnosis-oriented sensor optimization has been conducted mainly at single-machine level rather than at system level, i.e., the variation sources and locations of sensors are limited to a single manufacturing station [11]–[14]. The main approach used in single-station optimization is to optimize the Fisher Information Matrix (FIM) [15], denoted as $\mathbf{M}$, to ensure minimum estimation error. Given the sensor number, Fadale *et al.* [12] determined optimal sensor locations by maximizing $\mathrm{Det}(\mathbf{M})$, where $\mathrm{Det}(\cdot)$ is the determinant of a matrix. Khan *et al.* [13] determined sensor locations by maximizing the minimum distance among variation pattern vectors $\mathbf{d}(i)$'s. Udwadia [11] discussed the issues of how to place the given $m$ sensors; and, how to place additional $s$ sensors when $m$ sensors are already installed, in order to maximize $\mathrm{Det}(\mathbf{M})$ or minimize trace $((\mathbf{M}^T\mathbf{M})^{-1})$. Wang and Nagarkar [14] used a prediction matrix $\mathbf{H}$ in combination with $\mathbf{M}$, where $\mathbf{H}$ is the measure of the relative contribution of a sensor to the information provided by the sensor set. The sensor with the least contribution will be removed to reduce redundancy. The algorithm stops before there is a substantial decrease in $\mathrm{Det}(\mathbf{M})$.

Research on sensor distribution for multistation systems, which considers the effectiveness of variation diagnosis, is very limited. Khan *et al.* [16] and Khan and Ceglarek [17] studied (1) "end-of-line sensing," where the sensing station is located at the end of a manufacturing system, but variation sources include those from upstream stations, and (2) "distributed sensing," where sensing stations can be located in preselected yet arbitrary places in a manufacturing system. Their approach optimized sensor layout by maximizing the minimum distances between any pair of variation patterns, which were obtained using the variation model of a single fixture. The final sensor layout was obtained by aggregating variation patterns for all fixtures in the system. Their methodology does not consider the interaction effects of variation sources between different stations, which significantly impact the variation propagation in assembly systems and need to be taken into consideration. Table I presents the relationship between the proposed methodology and those available in the literature.

There are three critical challenges in achieving optimal sensor distribution in multistation systems: (1) sensing information obtained at various sensing stations needs to be integrated and effectively utilized; (2) a quantitative diagnosability measure is needed to benchmark the effectiveness of a sensor system; and (3) based on the integrated sensing information and the quantitative diagnosability measure, an optimal strategy needs to be developed.

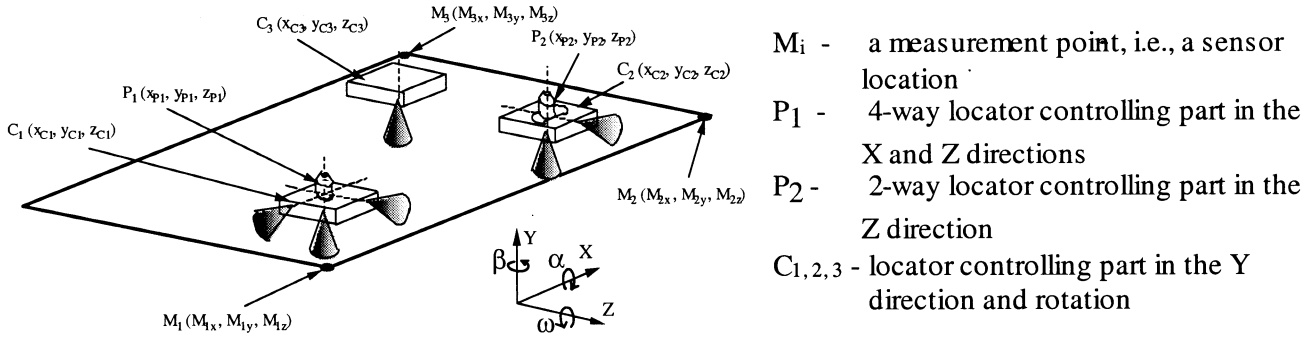


Fig. 1. A layout of 3-2-1 fixture with marked Ps, Cs and Ms points.

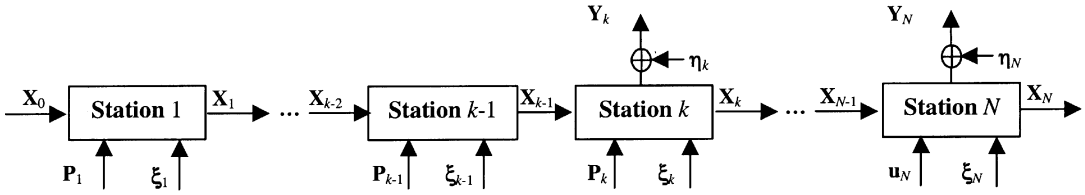


Fig. 2. Information flow in multistation manufacturing.

The first two challenges have been addressed in the authors' earlier publications—the sensing information on different stations is integrated and coordinated in a station-indexed state-space model [19], [20], and, a diagnosability index is developed to quantify the effectiveness of a distributed sensor system [21]. This paper focuses on presenting an optimal strategy for sensor distribution in a multistation assembly process, which essentially answers the two aforementioned problems (Q1) and (Q2). Problem (Q3) is then addressed using methods presented in [11]–[14].

Following this introduction, Section II briefly reviews the state space variation model and the development of the diagnosability index since they are integral parts of the development of an optimal sensing strategy. Section III presents the optimal strategy for sensor distribution and justifies the minimum number of sensing stations and sensors. A multistation assembly system is presented in Section IV to illustrate the optimal sensing strategy. Finally, the paper is summarized in Section V.

## II. Process Variation Model and Diagnosability Analysis

Diagnosis-oriented sensor optimization, as presented in this paper, often requires modeling of a physical manufacturing system with specific domain(s) of variation sources. In this paper, we focus on diagnosing dimensional variation sources in a multistation assembly process. The variation domain is limited to variation sources related to fixture failure (hereinafter referred to as "fixture variation"), which was identified as the major contributor of variation in dimensional quality control in assembly processes [22], [23].

A typical 3-2-1 fixture locating layout with five locators ($P_1$, $P_2$, $C_1$–$C_3$) and three measurement points ($M_1$–$M_3$) is shown in Fig. 1. In a multistation assembly process, fixture locators are key functional elements, providing parts support with Cartesian coordinates at each assembly station. A sensor system deployed in a multistation process aims to identify fixture failure by utilizing dimensional measurements of the finished product and/or of intermediate subassemblies.

The relationship between product measurements (e.g., measurements at points $M_1$–$M_3$ in Fig. 1) and fixture variation was modeled in a state space representation for multistation assembly processes [19], [20]. The basic idea is to consider a multistation process as a sequential system but replace the time index in a traditional state space model with a station index. For the process in Fig. 2, the station-indexed state space model can be expressed as

$$\mathbf{X}_k = \mathbf{A}_{k-1}\mathbf{X}_{k-1} + \mathbf{B}_k\mathbf{P}_k + \boldsymbol{\xi}_k$$

and

$$\mathbf{Y}_k = \mathbf{C}_k\mathbf{X}_k + \boldsymbol{\eta}_k \quad k \in \{1, 2, \ldots, N\} \tag{1}$$

where $k$ is the station index and $N$ is the number of stations. The product dimensional state, which describes dimensional deviations that occur randomly, is denoted as $\mathbf{X}_k$. Let $\mathbf{X}_{i,k} \equiv [\delta x_{i,k}\ \delta y_{i,k}\ \delta z_{i,k}\ \delta\alpha_{i,k}\ \delta\beta_{i,k}\ \delta\omega_{i,k}]^T$ (refer to Fig. 1 for the six coordinate variables) be the random deviations associated with each of the six degrees of freedom of part $i$ at station $k$, where $\delta$ is the deviation operator. Then the state of the product, which comprises $n_p$ parts, is represented by $\mathbf{X}_k \equiv [\mathbf{X}_{1,k}^T \cdots \mathbf{X}_{n_p,k}^T]^T$. If part $i$ has not yet appeared on station $k$, the corresponding $\mathbf{X}_{i,k} = \mathbf{0}$. The input vector $\mathbf{P}_k$ represents the random deviations associated with fixture locators on station $k$. Mathematically, fixture variations are characterized by the variances of random variables in $\mathbf{P}_k$, which are called the *variance components* of fixture variation. Additional process errors including un-modeled higher order terms are represented by $\boldsymbol{\xi}_k$. Product measurements at station $k$ are included in $\mathbf{Y}_k$. For the example in Fig. 1, $\mathbf{Y}_k$ is $[\delta M_{1x}\ \delta M_{1y}\ \delta M_{1z} \vdots \cdots \vdots \cdots\ \delta M_{3z}]^T$, i.e., the deviations associated with measurement features $M_1$, $M_2$, and $M_3$. But $\mathbf{Y}_k$ is not necessarily measured—if it so happens, then

TABLE II
INTERPRETATION OF SYSTEM MATRICES

| Symbol | Name | Relationship | Interpretation | Assembly Task |
|---|---|---|---|---|
| $\mathbf{A}$ | Dynamic matrix | $\mathbf{X}_{k-1} \xrightarrow{\mathbf{A}_{k-1}} \mathbf{X}_k$ | Change of fixture layout between two adjacent stations | Assembly transfer |
| $\mathbf{\Phi}_{k,i}$ | State transition matrix | $\mathbf{X}_i \xrightarrow{\mathbf{\Phi}_{k,i}} \mathbf{X}_k$ | Change of fixture layout among multiple stations | Assembly transfer |
| $\mathbf{B}$ | Input matrix | $\mathbf{P}_k \xrightarrow{\mathbf{B}_k} \mathbf{X}_k$ | Fixture layout at station $k$ | Part positioning |
| $\mathbf{C}$ | Observation matrix | $\mathbf{X}_k \xrightarrow{\mathbf{C}_k} \mathbf{Y}_k$ | Sensor layout at station $k$ | Inspection |

$\mathbf{C}_k = \mathbf{0}$. Sensor noise, denoted by $\boldsymbol{\eta}_k$, is a vector of uncorrelated random variables with zero means.

In the state space model, $\mathbf{A}_k$ and $\mathbf{B}_k$ include process design information such as fixture layouts at each station, as well as the effect of fixture layout change across stations. Matrix $\mathbf{C}_k$ includes sensor deployment information (the number and location of sensors on station $k$). The sensing information across different stations is integrated through this state space model. The corresponding physical interpretation of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is presented in Table II, where $\mathbf{\Phi}_{k,i} \equiv \mathbf{A}_{k-1}\mathbf{A}_{k-2}\cdots\mathbf{A}_i$ and $\mathbf{\Phi}_{i,i} \equiv \mathbf{I}$.

A detailed diagnosability analysis of a multistation system modeled by (1) was reported in [21]. We summarize the results that will be used in this paper. The recursive expression in (1) can be formulated into an input–output relation as

$$\mathbf{Y} = \bar{\mathbf{\Gamma}} \cdot \mathbf{P} + \bar{\mathbf{\Gamma}}_0 \cdot \mathbf{X}_0 + \boldsymbol{\varepsilon} \tag{2}$$

where $\mathbf{Y}^T \equiv [\mathbf{Y}_1^T\ \mathbf{Y}_2^T\ \cdots\ \mathbf{Y}_N^T]$, $\mathbf{P}^T \equiv [\mathbf{P}_1^T\ \mathbf{P}_2^T\ \cdots\ \mathbf{P}_N^T]$, $\boldsymbol{\varepsilon}^T \equiv [\boldsymbol{\varepsilon}_1^T\ \boldsymbol{\varepsilon}_2^T\ \cdots\ \boldsymbol{\varepsilon}_N^T]$, $\boldsymbol{\varepsilon}_k \equiv \Sigma_{i=1,k}\mathbf{C}_k\mathbf{\Phi}_{k,i}\boldsymbol{\xi}_i + \boldsymbol{\eta}_k$, and

$$\bar{\mathbf{\Gamma}} \equiv \begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix},$$

$$\bar{\mathbf{\Gamma}}_0 \equiv \begin{bmatrix} \mathbf{C}_1\mathbf{\Phi}_{1,0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix}. \tag{3}$$

Since we are concerned with the variation of state variables and input vectors, the original model (2) for random deviations is converted into a variation model. Assuming that product deviation $\mathbf{X}_0$, fixture deviation $\mathbf{P}_k$, and noise term $\boldsymbol{\varepsilon}_k$ are independent, we can have

$$\mathbf{\Sigma}^Y = \mathbf{\Gamma} \cdot \mathbf{\Sigma}^P \cdot \mathbf{\Gamma}^T + \mathbf{\Gamma}_0\mathbf{\Sigma}_0^X\mathbf{\Gamma}_0^T + \mathbf{\Sigma}^\varepsilon. \tag{4}$$

In (4), $\mathbf{\Sigma}_0^X$ is known from measurements obtained at the end of the precedent fabrication process. We also assume that $\mathbf{\Sigma}^\varepsilon$ can be estimated using data from a normal process condition when no outstanding fixture error occurs. With this assumption, $\widetilde{\mathbf{\Sigma}}$ is defined as the summation of all measured or estimable quantities $\mathbf{\Sigma}^Y - \mathbf{\Gamma}_0\mathbf{\Sigma}_0^X\mathbf{\Gamma}_0^T - \mathbf{\Sigma}^\varepsilon$. Then (4) can be simplified as

$$\widetilde{\mathbf{\Sigma}} = \mathbf{\Gamma} \cdot \mathbf{\Sigma}^P \cdot \mathbf{\Gamma}^T. \tag{5}$$

*Definition:* The fixture variation in a multistation assembly system is diagnosable if the variance components in $\mathrm{diag}(\mathbf{\Sigma}^P)$ can be uniquely determined, given the known/measured quantity $\widetilde{\mathbf{\Sigma}}$, where $\mathrm{diag}(\cdot)$ extracts the diagonal elements from a matrix into a vector.

A derivation in [21] shows that (5) can be expressed as

$$\mathrm{vec}(\widetilde{\mathbf{\Sigma}}) = \pi(\mathbf{\Gamma}) \cdot \mathrm{diag}(\mathbf{\Sigma}^P) \tag{6}$$

where $\mathrm{vec}(\cdot)$ is the vector operator [24], $\pi(\cdot)$ is a matrix transformation defined as

$$\pi(\mathbf{\Gamma}) \equiv \Big[(\boldsymbol{\gamma}^{1*}\boldsymbol{\gamma}^1)^T \quad \cdots \quad (\boldsymbol{\gamma}^{1*}\boldsymbol{\gamma}^q)^T \vdots (\boldsymbol{\gamma}^{2*}\boldsymbol{\gamma}^1)^T \quad \cdots \quad (\boldsymbol{\gamma}^{2*}\boldsymbol{\gamma}^q)^T \vdots \cdots \vdots (\boldsymbol{\gamma}^{q*}\boldsymbol{\gamma}^q)^T\Big]^T \tag{7}$$

and $\boldsymbol{\gamma}^k$ is the $k$th row vector of $\mathbf{\Gamma}$, "*" represents a Hadamard product [24].

Define matrix $\mathbf{D}_N$ as

$$\mathbf{D}_N \equiv \pi(\mathbf{\Gamma}) = \pi\left(\begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix}\right) \tag{8}$$

and $\mathbf{D}_N$ is called the diagnosability matrix. The diagnosable condition of fixture variations as stated in [21] is that $\mathbf{D}_N^T\mathbf{D}_N$ should be of full rank, which is equivalent to that $\mathbf{D}_N$ is of full column rank. Meanwhile, a diagnosability index $\mu$ is defined as

$$\mu = \frac{\rho(\mathbf{D}_N)}{\sum_{k=1}^N m_k} \tag{9}$$

where $\rho(\cdot)$ is the rank of a matrix and $m_k$, namely the dimension of $\mathbf{P}_k$, is the number of variance components in $\mathrm{diag}(\mathbf{\Sigma}^P)$. The $\mu$ is a normalized quantity in [0, 1]. The condition of $\mu = 1$ is equivalent to $\mathbf{D}_N^T\mathbf{D}_N$ being of full rank. Thus, we say that a

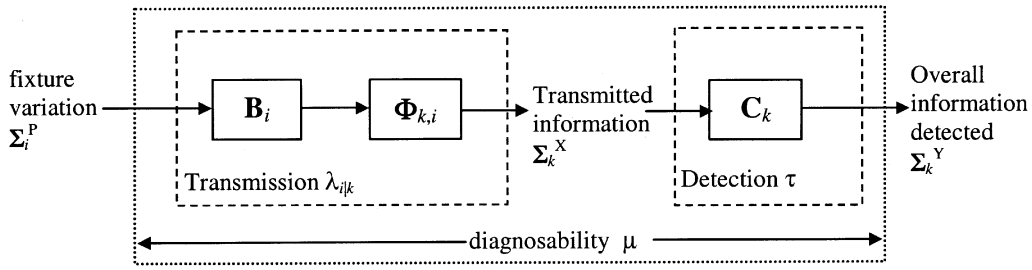


Fig. 3. Variation transmission and detection.

sensor system provides the complete diagnosability if and only if $\mu = 1$. We would like to make the following remarks.

*Remark 2.1:* If $\mathbf{\Sigma}^{\varepsilon}$ will have to be estimated on-line using production data, a new diagnosability condition should be developed. In fact, such a condition was recently developed in [25]. It depends on the structure of system matrices $\mathbf{A}$ (or $\mathbf{\Phi}$), $\mathbf{B}$, and $\mathbf{C}$ whether or not the noise covariance $\mathbf{\Sigma}^{\varepsilon}$ can be separated from fixture variations.

*Remark 2.2:* Diagnosability is based on the concept of the rank of a matrix. Determining the rank of a matrix is a difficult problem from a numerical perspective. However, with the help of a mathematical software package such as MATLAB, the difficulty associated with computing a matrix rank has been considerably alleviated. Similar rank-related conditions are commonly used in engineering applications, e.g., the controllability and observability conditions in control theory [26].

*Remark 2.3:* Based on the above diagnosable condition, any $\mu < 1$ indicates that fixture variation is only partially diagnosable. A single index $\mu$ is insufficient to characterize a partially diagnosable system, of which the preferable diagnosability condition may also depend on how nondiagnosable variation sources are coupled. An analysis dedicated to partially diagnosable systems is presented in [25] and is not repeated here. But, for a sensor distribution problem, we should always try to achieve full diagnosability, unless other evidence or engineering experiences indicate that those nondiagnosable variation sources are not of practical concern.

## III. Optimal Sensor Distribution Strategy

### A. Sensing Cost and Objective Function

The objective of an optimal sensor-distribution strategy is to achieve the desired diagnosability at a minimum cost. The cost of a sensor system comes not only from sensors but also from the expense of building sensing stations. It is assumed that all parts assembled at any station can be physically accessed by sensors at a downstream station for their positional and orientation measurements. Thus, there are two ways of measuring product features during production: 1) sensors are installed directly on the assembly station and measurements are taken after the assembly operation is completed; or 2) the subassembly is transferred to a dedicated station designed solely for taking measurements. In both cases, these stations are called sensing stations. A sensing station will incur extra cost when compared to a regular assembly station. This is clear for the second case with a dedicated sensing station. However, this is also true for the first case due to special requirements for upgrading a regular station to install sensors. The specific monetary cost is usually different for both cases. However, the cost differences between the two cases are not considered in this paper.

In order to diagnose all fixture variations, it is required that $\mu = 1$ (9). With $\mu = 1$ as the constraint, the optimization scheme can be formulated as

$$J_{\text{opt}} = \min \left\{ c_1 \cdot \sum_{k=1}^{N} (\#\text{ of sensors at station } k) + c_2 \cdot (\#\text{ of sensing stations}) \right\} \quad \text{subject to } \mu = 1 \qquad (10)$$

where $c_1$ and $c_2$ are the average cost per sensor and per sensing station, respectively.

In this section, we will decompose the system-wide diagnosability in a multistation process into two steps (Fig. 3): 1) the transmission of variation from station $i$ to station $k$, with the transmitted information modeled by state covariance matrix $\mathbf{\Sigma}_k^X$ and 2) the detection of fixture variation by sensors located at station $k$, with the overall information modeled by a measurement covariance matrix $s$. Information transformation in the two steps is characterized by the transmissibility ratio $\lambda_{i|k}$ and the detecting power $\tau_k$ (on station $k$), respectively. The optimal sensor distribution is studied through: 1) achieving the optimal detecting power on a single station; and 2) identifying stations at which error information is not completely transmitted (i.e., $\lambda_{i|k} < 100\%$). The detailed development is organized as follows: Section III-B defines $\lambda_{i|k}$ and derives its properties for a multistation assembly process. Ratio $\lambda_{i|k}$ will be used to determine where to build a sensing station. Section III-C studies the effect of sensor layout at an individual station on its detecting power $\tau_k$, which leads to several practical rules for placing sensors on a single sensing station. The optimal sensor-distribution strategy given in Section III-D is a natural outcome of results from Sections III-B and III-C.

### B. Variation Transmissibility Ratio

When the transmission of variation is studied, we assume that a sufficient number of sensors are installed at station $k$. We will further discuss the meaning of "a sufficient number of sensors" in Section III-C. For the time being, let us assume it to mean $\mathbf{C}_k = \mathbf{I}$.

We notice that variation transmission is determined by process configuration such as fixture layout geometry (modeled by $\mathbf{B}_i$) and the change in fixture layouts between stations (modeled by $\mathbf{\Phi}_{k,i}$). The 3-2-1 fixture shown in Fig. 1 can restrain DOF degrees of freedom (d.o.f.) of a rigid workpiece (a workpiece could be a single part or a multipart subassembly),

where $\mathrm{DOF} = 3$ for a 2-D workpiece and $\mathrm{DOF} = 6$ for a 3-D workpiece. Suppose that there are $p_i$ '3-2-1' fixtures on station $i$ and each of them supports one rigid workpiece. The total number of d.o.f. that these fixtures restrained is $p_i \cdot \mathrm{DOF} = m_i = \mathrm{dimension}(\mathbf{P}_i)$, which is the number of independent variation sources associated with the $p_i$ fixtures. Thus $m_i$ is the number of unknown variance components of fixture variation that we try to diagnose. On the other hand, $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i))$ represents the number of independent equations that we have in solving the $m_i$ unknown variance components. If $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)) < m_i$, not all variance components of fixture variation on station $i$ can be uniquely solved. In that case, some information regarding fixture variation on station $i$ is lost during the transmission step. We define a transmissibility ratio $\lambda_{i|k}$ to quantify the variation transmission from station $i$ to station $k$ as

$$\lambda_{i|k} \equiv \frac{\rho\left(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)\right)}{m_i} \tag{11}$$

where $\lambda_{i|k} = 1$ suggests that the complete information regarding fixture variation has been transmitted from station $i$ to station $k$. If loss of information occurs during the transmission step, $(1-\lambda_{i|k})$ is used to quantify the information loss. Any information loss during the transmission step suggests that fixture variations at station $i$ are not fully diagnosable regardless of the number of sensors placed on station $k$. Furthermore, we have

*Lemma 1:* A transmissibility ratio possesses the following properties:

$$(P1)\lambda_{i|i} = 1;\ (P2)\lambda_{i|j} = \lambda_{i|k} \geq \frac{p_i - 1}{p_i}, \qquad k > j > i.$$

The proofs of both properties are presented in Appendix I. The first property is intuitive because it implies that if we measure all the dimensional information of a workpiece (say, let $\mathbf{C}_i = \mathbf{I}$), the variation of the fixture that is currently used to support the workpiece can be uniquely determined. The second property seems counterintuitive. It says that the variation transmissibility from station $i$ to station $j(j > i)$ is the same as that from station $i$ to station $k$ which is located further downstream; $k > j > i$. This is an important property describing transmission of fixture variation in a multistation assembly process, under the condition that all measurement points on a product/part can be measured at any station if needed.

In [21], the diagnosability $\mu$ is further partitioned into within-station and between-station diagnosability, respectively. Generally, a system is fully diagnosable only if it is both within-station and between-station diagnosable. However, for a multistation assembly process where new parts are added on downstream stations, it is possible to ensure between-station diagnosability through end-of-line sensing. This conclusion is stated in Lemma 2 and Corollary 1. The proof of Lemma 2 is included in Appendix II. Corollary 1 is a straightforward result from Lemma 2 and its proof is thus omitted.

*Lemma 2:* For a multistation assembly process where new parts are added on downstream stations, given that a sufficient number of sensors are installed on station $N$ (to measure the final product), the between-station diagnosability condition will be satisfied.

*Corollary 1:* Under the same condition as Lemma 2, the diagnosability $\mu$ can be expressed as

$$\mu = \frac{\sum_{i=1}^{N} \lambda_{i|N} \cdot m_i}{\sum_{i=1}^{N} m_i}. \tag{12}$$

*Remark 3.1:* The index $\lambda_{i|k}$ is solely determined by fixture design configuration and, thus, it can be calculated after the process is designed but before the sensor positions are allocated. The values of $\lambda_{i|k}$'s are not modifiable after the process design phase is completed. We will utilize Lemmas 1 and 2 to decide at which station(s) to place sensors to retrieve the information lost during the transmission step.

*Remark 3.2:* From (12), $\mu$ will be 1 if all $\lambda_{i|N}$'s are 1, i.e., fixture variations on all upstream stations are diagnosable by taking measurements on station $N$. In such a case, we need install sensors only on the last station $N$. In many cases, not all $\lambda_{i|N}$'s are equal to 1. If $\lambda_{i|N} < 1$, the strategy of increasing transmissibility by installing sensors on any stations between $i+1$ and $N-1$ will not help, since $\lambda_{i|k} = \lambda_{i|N} < 1$ for $k = i+1, \ldots, N-1$, according to (P2) in Lemma 1. The only recommended solution is to add sensors directly on station $i$ since $\lambda_{i|i} = 1$ (P1 in Lemma 1). The same procedure will be repeated for all stations with $\lambda_{i|N} < 1$.

*Remarks 3.3:* As indicated in Fig. 3, the actual information retrieved on station $k$ will also depend on the number and layout of sensors on station $k$ (modeled by $\mathbf{C}_k$). Ideally, if there is no restriction on sensor number, we can place "a sufficient number of sensors" to make $\mathbf{C}_k = \mathbf{I}$ so that all information transmitted to $\mathbf{\Sigma}_k^X$ is retrievable. However, in order to minimize the sensing cost, we need to find the most economical sensor layout at station $k$, which places the *minimum* number of sensors but retrieves the complete transmitted information in $\mathbf{\Sigma}_k^X$.

### C. Sensor Placement on a Single Station

Suppose that sensors are installed on station $N$ with the resultant diagnosability of $\mu$. The total number of variance components to be diagnosed from station 1 to $N$ is $\mathbf{\Sigma}_{i=1}^{N} m_i$. Then, the quantity of $\mu \cdot \mathbf{\Sigma}_{i-1}^{N} m_i$ is considered to be the amount of information retrieved by sensors on station $N$, and $\mathbf{\Sigma}_{i=1}^{N} \lambda_{i|N} \cdot m_i$ represents the amount of information of fixture variation transmitted from upstream stations. If $\mathbf{C}_k$ is assumed to be $\mathbf{I}$, then $\mu \cdot \mathbf{\Sigma}_{i=1}^{N} m_1$ always equals to $\mathbf{\Sigma}_{i=1}^{N} \lambda_{i|N} \cdot m_i$ (Corollary 1). However, given an arbitrary number and layout of sensors, we may have the inequality $\mu \cdot \mathbf{\Sigma}_{i=1}^{N} m_i < \mathbf{\Sigma}_{i=1}^{N} \lambda_{i|N} \cdot m_i$. Then, we define a detectability ratio $\tau$ as

$$\tau = \frac{\mu \cdot \left(\sum_{i=1}^{N} m_i\right)}{\sum_{i=1}^{N} \lambda_{i|N} \cdot m_i}. \tag{13}$$

Sensor placement on an individual station is considered as having a sufficient number of sensors if variation detectability $\tau = 1$. The minimum sufficient sensor number is studied for two cases: deviation detection and variation detection.

*1) The Minimum Sufficient Number of Sensors in Detecting Product Positional Deviations:* In an assembly process, coordinate sensors are used to measure the positional deviation of a rigid workpiece. One coordinate sensor can measure 3 d.o.f. on

a 3-D workpiece or 2 d.o.f. on a 2-D workpiece. In order to measure all d.o.f. of a workpiece, three independent sensors are required for a 3-D case (or two sensors for a 2-D case). If all d.o.f. of each part in an assembly are measured, it can be concluded that the positional deviations of the product in all directions have been obtained. Recall that there are $n_p$ parts in the assembly on station $N$. Then, we need $3n_p$ (or $2n_p$ for 2-D) sensors with three sensors on every part to detect positional deviations of all parts. Once positional deviations of all parts are detected, we can calculate $\mathbf{\Sigma}_N^X$ from the deviational measurements. It is not true the other way around, though. That is, even if we know $\mathbf{\Sigma}_N^X$, we cannot reconstruct the deviational measurements. Thus, the condition for detecting the deviations of all parts is a sufficient condition for variation detection required in (13). In the case of variation detection, it is possible to reduce the number of sensors while still reaching $\tau = 1$.

*2) The Minimum Sufficient Number of Sensors in Detecting Product Positional Variations:* The reasoning behind a possible sensor-number reduction lies in the application of the $\pi$-transform in the variation model (6). Suppose that we have a deviation relationship represented as $\mathbf{Y} = \mathbf{\Gamma P}$. The d.o.f. of parts is the same as the dimension of $\mathbf{P}$ if all parts are completely restrained. In order to solve for fixture deviations, the dimension of $\mathbf{Y}$ has to be at least the same as that of $\mathbf{P}$ and $\mathbf{\Gamma}^T\mathbf{\Gamma}$ should be of full rank. When fixture variations of $\mathbf{P}$ are considered, the model becomes $\mathrm{vec}(\mathbf{\Sigma}^Y) = \pi(\mathbf{\Gamma}) \cdot \mathrm{diag}(\mathbf{\Sigma}^P)$, according to (6). Since fixture deviations in $\mathbf{P}$ are physically independent, $\mathbf{\Sigma}^P$ is a diagonal matrix. The $\mathrm{diag}(\mathbf{\Sigma}^P)$, which contains all variance components of fixture variation, is of the same size as $\mathbf{P}$. This suggests that the number of unknowns is not changed from that of the deviation model, but the number of known quantities in $\mathrm{vec}(\mathbf{\Sigma}^Y)$ increases compared to the number of elements in $\mathbf{Y}$. If $\mathbf{Y}$ is of $q \times 1$, $\mathrm{vec}(\mathbf{\Sigma}^Y)$ is then of size $q(q+1)/2 \times 1$. These additional terms in $\mathrm{vec}(\mathbf{\Sigma}^Y)$ are the covariances between the variables in $\mathbf{Y}$. The $\pi$-transform takes this change into account so that $\pi(\mathbf{\Gamma})$ increases the number of rows but keeps the number of columns the same as that of $\mathbf{\Gamma}$. Based on the fact that the covariance terms in the variation model provide more known quantities, the required number of sensors can be reduced. Let $w$ be the dimension of $\mathbf{P}$. As for the deviation model, the realization of diagnosability requires that $q \geq w$ and $\mathbf{\Gamma}^T\mathbf{\Gamma}$ be of full rank. But for the variation model, it requires that $q(q+1)/2 \geq w$ and $\pi(\mathbf{\Gamma})^T\pi(\mathbf{\Gamma})$ be of full rank.

Let us illustrate this by using a simple example. Let $q = 2$, $w = 3$, and $\bar{\mathbf{\Gamma}}$ be

$$\bar{\mathbf{\Gamma}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{so} \quad \bar{\mathbf{\Gamma}}^T\bar{\mathbf{\Gamma}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \quad (14)$$

Apparently, we would not be able to solve for the deviation variables in $\mathbf{P}$ since $q < w$ and $\bar{\mathbf{\Gamma}}^T\bar{\mathbf{\Gamma}}$ is singular. The $\pi(\bar{\mathbf{\Gamma}})$ turns out to be

$$\pi(\bar{\mathbf{\Gamma}}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \pi(\bar{\mathbf{\Gamma}})^T\pi(\bar{\mathbf{\Gamma}}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (15)$$

Here, $\pi(\bar{\mathbf{\Gamma}})^T\pi(\bar{\mathbf{\Gamma}})$ is of full rank and $q(q+1)/2 = 3 = w$. Thus, we can solve for the variations of $\mathbf{P}$. If we write down $\mathbf{\Sigma}^Y$ in terms of the variance components in $\mathrm{diag}(\mathbf{\Sigma}^P)$, it is

$$\mathbf{\Sigma}^Y = \begin{bmatrix} \sigma_{P_1}^2 + \sigma_{P_2}^2 & \sigma_{P_2}^2 \\ \sigma_{P_2}^2 & \sigma_{P_2}^2 + \sigma_{P_3}^2 \end{bmatrix} \quad (16)$$

where $\sigma_{P1}^2$, $\sigma_{P2}^2$, and $\sigma_{P3}^2$ are three variance components in $\mathrm{diag}(\mathbf{\Sigma}^P)$. From (16), we verify that the variance components in $\mathrm{diag}(\mathbf{\Sigma}^P)$ cannot be uniquely solved from the diagonal variance terms in $\mathbf{\Sigma}^Y$. The covariance term between $y_1$ and $y_2$ lets $\mathbf{Y} = [y_1\ y_2]^T$) equal $\sigma_{P2}^2$, providing an additional equation, with which all three variance components can be uniquely determined.

We have explained why the number of sensors can be reduced for variation diagnosis. However, an important question remains to be answered: how many sensors are necessary to reach $\tau = 1$? The minimum number of sensors satisfying $\tau = 1$ depends on how the sensors are allocated on different parts. For instance, given a subassembly consisting of two parts, we can place six sensors on one of the two parts, but the information they provide will be the same as that provided by three sensors placed on the same part. A more efficient way is to place three sensors on each of the two parts, respectively.

The sensor placement on an individual station is modeled by $\mathbf{C}_k$. In general, it is very difficult to show analytically how the sensor placement will affect the rank of $\pi(\mathbf{\Gamma}_i)$, where $\mathbf{\Gamma}_i = \mathbf{C}_k\mathbf{\Phi}_{k,i}\mathbf{B}_i$ (refer to the information chain in Fig. 3). Therefore, we conduct a numerical study to produce certain practical rules that we can follow. In the numerical study, for the sake of simplicity, we consider a two-station assembly process, which consists of all assembly operations (such as part positioning, joining, and transferring) so that it captures the interactions in $\mathbf{C}_k\mathbf{\Phi}_{k,i}\mathbf{B}_i$. Sensors will be placed on the second station in this two-station assembly segment.

The numerical test results are summarized in Table III, where $\tau(s_1, \ldots, s_j)$ is the detectability when $s_1, \ldots, s_j$ sensors are placed on part 1, part 2,. . ., part $j$, respectively. There are a limited number of options in placing these sensors. The maximum number of sensors is $3n_p$ (3-D) or $2n_p$ (2-D) for all product-deviations to be made detectable. For instance, if a subassembly consists of two 2-D parts, the maximum number of sensors needed is four. The possible sensor placements constitute the following sets: (0,4), (4,0), (1,3), (3,1), or (2,2). For each sensor placement, we test detectability $\tau$ through a numerical calculation. Comparison among all the possible sensor placements will lead us to the minimum sufficient number of sensors and the associated scheme of sensor placement. It should be noticed that a small position change of sensor locations on the same part may not affect diagnosability $\mu$ defined in (9). In this numerical test, the position of each sensor is determined by following a simplified procedure, which postulates that no two sensors can be located at the same position and no positions of any three sensors can be collinear. We observe the following.

(C1) Given the same number of sensors, the detectability is larger if the sensors are placed on different parts in a rigid multipart subassembly than if they are placed on the same part.

TABLE III
NUMERICAL ANALYSIS OF SENSOR PLACEMENT ON INDIVIDUAL STATION

| # of parts | # of sensors | Sensor placements and τ |
|---|---|---|
| $n_p$=2 | 1 | τ(0,1)=τ(1,0)=0.43; |
| | 2 | τ(0,2)=τ(2,0)=0.43; **τ(1,1)=1**; |
| | 3 | τ(0,3)=τ(3,0)= 0.43; τ(1,2)=τ(2,1)=1; |
| | 4 | τ(0,4)=τ(4,0)= 0.43; τ(1,3)=τ(3,1)=1; τ(2,2)=1; |
| $n_p$=3 | 1 | τ(1,0,0)=τ(0,1,0)=τ(0,0,1)=0.25; |
| | 2 | τ(2,0,0)=τ(0,0,2)=0.25; τ(0,2,0)=0.33; τ(1,0,1)=0.58; τ(1,1,0)=0.67; τ(0,1,1)=0.75; |
| | 3 | τ(3,0,0)=τ(0,0,3)=0.25; τ(0,3,0)=0.33; τ(2,0,1)=τ(1,0,2)=0.58; τ(1,2,0)= τ(2,1,0)=0.67; τ(0,1,2)=τ(0,2,1)=0.75; **τ(1,1,1)=1**; |
| | 4 | τ(4,0,0)=τ(0,0,4)=0.25; τ(0,4,0)=0.33; τ(3,0,1)=τ(1,0,3)=τ(2,0,2)=0.58; τ(1,3,0)=τ(3,1,0)=τ(2,2,0)=0.67; τ(0,1,3)=τ(0,3,1)=τ(0,2,2)=0.75; τ(1,1,2)=τ(2,1,1)=τ(1,2,1)=1; |
| | 5 | only for τ=1. τ(1,1,3)=τ(3,1,1)=τ(1,3,1)=τ(2,2,1)=τ(2,1,2)=τ(1,2,2)=1 |
| | 6 | only for τ=1. τ(1,1,4)=τ(4,1,1)=τ(1,4,1)=τ(3,2,1)=τ(2,3,1)=τ(3,1,2)= τ(2,1,3)=τ(1,3,2)=τ(1,2,3)=τ(2,2,2)=1 |
| $n_p$=4 | 4 | τ(4,0,0,0)=τ(0,0,0,4)=0.20; τ(0,4,0,0)=τ(0,0,4,0)=0.27; τ(1,0,0,3)= τ(3,0,0,1)=0.47; τ(1,3,0,0)=τ(3,1,0,0)=τ(1,0,3,0)=τ(3,0,1,0)=0.53; τ(0,1,3,0)=τ(0,3,1,0)=τ(0,0,1,3)=τ(0,0,3,1)=τ(0,1,0,3)=τ(0,3,0,1)=0.6; τ(1,1,2,0)=τ(1,1,0,2)=τ(1,0,1,2)=τ(1,0,2,1)=τ(1,2,1,0)=τ(1,2,0,1)= τ(0,1,1,2)=τ(0,1,2,1)= τ(0,2,1,1)=τ(2,,0,1,1)=τ(2,1,0,1)=τ(2,1,1,0)=0.8; **τ(1,1,1,1)=1**; |
| | … | cases with the number of sensors = 5, 6, 7, 8 were omitted. |

(C2) In order to make $\tau = 1$, at least one sensor should be placed on each part.

(C3) The minimum sufficient number of sensors is $n_p$ and the associated distribution is one sensor per part.

In Table III, we list only the detectability values up to $n_p = 4$. This type of exercise can be continued for more parts with more sensors. Recall that the reason why the number of sensors can be reduced is due to the extra information generated from the covariance terms between variables in $\mathbf{Y}$. The fact that the number of covariance terms is a quadratic function of the number of variables in $\mathbf{Y}$ will gain us more information when additional parts are involved. Thus, when more parts are involved, the above conclusions shall hold true. We summarize this idea in the following Lemma.

*Lemma 3:* When each part in an assembly has the same number of degrees of freedom, sensors should be uniformly allocated among all parts within an individual station so that one sensor per part will make $\tau = 1$.

### D. Optimal Strategy of Sensor Distribution

Lemmas 1 and 2 in Section III-B indicate that, if a variation source can be diagnosed in the next station, it will be diagnosable in any subsequent station. On the other hand, if fixture variation is not diagnosable at the following station, sensors will have to be placed right on the station where the fixture error occurs; such a station will be indicated by a $\lambda_{i|k} < 100\%$. Meanwhile, Lemma 3 in Section III-C shows that if one wants to use the minimal number of sensors to detect all the transmitted variation in $\mathbf{\Sigma}_k^X$, one sensor should be allocated to each part. We use Table IV to summarize the meaning of $\mu$, $\lambda$, and $\tau$, as well as the above understandings regarding variation transmissibility and sensor placement on individual stations.

The following algorithm, which distributes sensors in the multistation assembly process, is a natural outcome of the results from Sections III-B and III-C.

TABLE IV
INTERPRETATION OF THREE INDEXES $\mu$, $\lambda$, AND $\tau$

| Symbol | Name | Interpretation | Impact |
|---|---|---|---|
| $\lambda_{ijk}$ | Transmissibility | Benchmarks how much information of fixture variation is transmitted from station *i* to station *k*. | Determine the locations of sensing stations. |
| τ | Detectability | Benchmarks how much information among which has been transmitted to station *k* can be detected by sensors installed on station *k*. | Determine the *minimum* sufficient number of sensors and the corresponding sensor layout on station *k*. |
| μ | Diagnosability | Characterize the overall information retrieved by a sensor system, $\mu=f(\lambda,\tau)$. | Combine the above two. |

*Strategy of Sensor Distribution:*

Step 1) On station $k = N$ (the last station), place one sensor on each part. If $\mu = 1$, then stop, else, go to Step 2.

Step 2) Let $k = k - 1$, given the installed sensors at all downstream stations $k+1, k+2, \ldots, N$, check if $\lambda_{k|N}$ equals to 1, install sensors on station with $\lambda_{k|N} \neq 1$. The installation procedure follows the general rule of sensor placement on an individual station. If $\lambda_{k|N} = 1$, then do not install any sensor on that station.

Step 3) Stop if $\mu = 1$, otherwise repeat Step 2 for $k = N-1, N-2, \ldots, 1$.

*Theorem:* The above sensor-distribution strategy attains the optimization objective in (10).

*Proof:* Let $I_k$ denote the number of parts (including the parts in a multipart subassembly) on station $k$, where $I_{k-1}$ of them are in the major subassembly from the previous station $k-1$ and $(I_k - I_{k-1})$ are the new parts added at station $k$. Let us compare the situations when sensors are installed at station $N$ and station $N-1$, respectively. When sensors are installed at station $N$, $I_{N-1}$ sensors are installed on the major subassembly coming from station $N-1$ to make $\tau = 1$, following Lemma 3 in Section III-C. Sensors are also placed on the remaining $(I_N - I_{N-1})$ number of parts that are newly added at station $N$. However, since our intention is to find on which station (i.e., $N-1$ or $N$) sensors should be placed so that the same amount of variation information can be detected using less number of sensors, we consider that a fair comparison is to exclude the number of those sensors on the $(I_N - I_{N-1})$ new parts, given the fact that there is no alternative way of measuring these parts other than placing sensors on station $N$.

The average information regarding fixture variation detected by a single sensor is

$$\frac{\sum_{i=1}^{N}\lambda_{i|N}\cdot m_i}{I_{N-1}} = \frac{\sum_{i=1}^{N-2}\lambda_{i|N}\cdot m_i + \lambda_{N-1|N}\cdot m_{N-1} + m_N}{I_{N-1}}. \tag{17}$$

If $I_{N-1}$ sensors are installed on station $N-1$ to measure the same $I_{N-1}$ number of parts to make $\tau = 1$, the average information regarding fixture variation detected by one sensor is

$$\frac{\sum_{i=1}^{N-1}\lambda_{i|N-1}\cdot m_i}{I_{N-2}+(I_{N-1}-I_{N-2})} = \frac{\sum_{i=1}^{N-2}\lambda_{i|N-1}\cdot m_i + m_{N-1}}{I_{N-1}}. \tag{18}$$

According to (P2) in Lemma 1, $\lambda_{i|N-1} = \lambda_{i|N}$, $i \in [1, N-2]$

$$\sum_{i=1}^{N-2}\lambda_{i|N}\cdot m_i \text{ in (17)} = \sum_{i=1}^{N-2}\lambda_{i|N-1}\cdot m_i \text{ in (18)}. \tag{19}$$

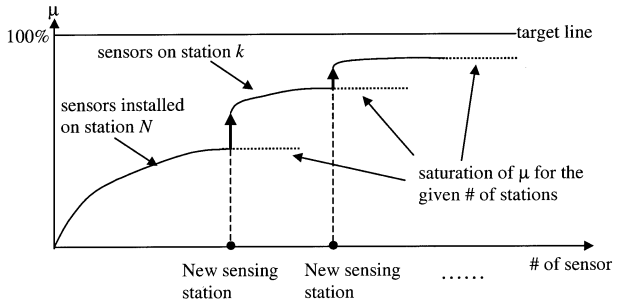


Fig. 4. Diagnosability property related to sensor distribution.

Notice that $\lambda_{N-1|N} \geq (p_{N-1}-1)/(p_{N-1})$ (again P2 in Lemma 1)

$$\begin{aligned}\therefore \lambda_{N-1|N}\cdot m_{N-1} &= \lambda_{N-1|N}\cdot p_{N-1}\cdot \mathrm{DOF} > (p_{N-1}-1)\cdot \mathrm{DOF} \\ &= m_{N-1}-\mathrm{DOF} \\ \therefore \text{Equation } (17) &\geq \frac{\sum_{i=1}^{N-2}\lambda_{i|N}\cdot m_i + m_{N-1}+m_N-\mathrm{DOF}}{I_{N-1}}.\end{aligned} \tag{20}$$

Also, $m_N = p_N \cdot \mathrm{DOF} > \mathrm{DOF}$, then, (17) > (18), which indicates that sensors installed on station $N$ are more effective than sensors installed on station $N-1$ in the sense of information detected per sensor. Following the mathematical induction approach, it can be shown that sensors installed on station $N-1$ are more effective than those on station $N-2$ and so on. We then conclude that the backward propagation strategy is an optimal strategy. □

We expect to see the change of diagnosability $\mu$, as shown in Fig. 4, when sensors are sequentially installed in a production line, which is elaborated as follows.

(I) When sensors are placed on station $N$, the entire transmitted information regarding fixture variation is detected and the system diagnosability increases rapidly with more sensors placed on station $N$. If $\lambda_{i|N} \neq 1$ for some station $i$, the maximum $\mu$ that can be achieved with sensors placed only on station $N$ is always smaller than 1, i.e., $\mu = (\boldsymbol{\Sigma}_{i=1}^{N}\lambda_{i|N}\cdot m_i)/(\boldsymbol{\Sigma}_{i=1}^{N} m_i) < 1$ [from (12)]. This is illustrated in Fig. 4 as a dotted flat line; we say that the diagnosability level is saturated.

(II) Thus, we should place some sensors at the upstream station(s) where $\lambda_{i|N} \neq 1$. Since $\lambda_{i|i} = 1$, the installation of sensors directly on station $i$ can help further increase the system diagnosability (step increases in diagnosability are seen in Fig. 4). The diagnosability obtained by sensors on station $N$ is $\lambda_{i|N}\cdot m_i$ (transmitted information), and the diagnosability obtained by sensors on station $i$ is $(1-\lambda_{i|N})\cdot m_i$ (information loss during transmission to station $N$). However, given that $\lambda_{i|N} \geq (p_i-1)/p_i$, $\lambda_{i|N} > (1-\lambda_{i|N})$ for $p_i > 2$, the slope of the curve (to the right) is less steep than the curve discussed in item (I). Again, the diagnosability will saturate at a higher level until sensors are installed on all stations where $\lambda_{i|N} \neq 1$. Following this procedure full diagnosability is achieved and $\mu = 1$.

## IV. Case Study

The optimal strategy of sensor distribution is illustrated by optimizing a sensor system in a three-station assembly process with four parts marked as 1, 2, 3, and 4 in Fig. 5. The assembly process can be described in three steps: 1) parts 1 and 2 are assembled at Station I [Fig. 5(a)]; 2) subassembly "1 + 2" is joined with parts 3 and 4 at Station II [Fig. 5(b)]; and 3) the final assembly is inspected for surface finish and welding quality at Station III [Fig. 5(c)]. A part or a subassembly is restrained by a fixture, which has a four-way pinhole locating pair that controls motion in both $x$ and $z$ directions; and, a two-way pinslot locating pair that controls motion only in the $z$ direction (only a 2-D case is presented in this example). After two parts are assembled, a subassembly still needs a four-way pin and a two-way pin to completely control its d.o.f. For example, subassembly "1 + 2" is positioned by the fixture locators $\{P_1, P_4\}$ that are on part 1 and part 2, respectively [Fig. 5(b)].

Parameters used in this example are $N = 3$; $\mathrm{DOF} = 3$; the number of fixtures on each station are $p_1 = 2$, $p_2 = 3$, and $p_3 = 1$. In such a 2-D case, $\mathbf{X}_{i,k} = [\delta x_{i,k}\ \delta z_{i,k}\ \delta\beta_{i,k}]^T$, which are deviations associated with two translational and one rotational d.o.f. of part $i$ on station $k$. The state equations of this three-station process are

$$\mathbf{X}_1 = \mathbf{B}_1\mathbf{P}_1 + \boldsymbol{\xi}_i$$

and

$$\mathbf{X}_k = \mathbf{A}_{k-1}\mathbf{X}_{k-1} + \mathbf{B}_k\mathbf{P}_k + \boldsymbol{\xi}_k, \quad k = 2,3 \tag{21}$$

where the initial state $\mathbf{X}_0$, representing part deviation from design nominals caused by the stamping process, is assumed negligible. Numerical expressions for $\mathbf{A}$ and $\mathbf{B}$ of the assembly process shown in Fig. 5 are given in (22) and (23), shown at the bottom of the next page. Based on them, the transmissibility ratios $\lambda_{i|N}$ are calculated as: $\lambda_{1|3} = 0.667$; $\lambda_{2|3} = 1$; $\lambda_{3|3} = 1$.

Step 1) On the last station $k = N = 3$, install one sensor on every part.

The value of $\mu$ is calculated given different numbers of sensors. The value of $\mu$ keeps increasing until it saturates (Curve-1 in Fig. 6) at the level of 0.889 for four sensors, one sensor per part. Further increase in the number of sensors on Station III does not increase the index $\mu$ (the dash line of Curve-1 in Fig. 6). The saturated value of $\mu$ can be computed by $\mu = (\sum_{i=1}^{3}\lambda_{i|3}\cdot p_i\cdot \mathrm{DOF})/(\sum_{i=1}^{3} p_i\cdot \mathrm{DOF}) = (0.667\cdot 2\cdot 3 + 1\cdot 3\cdot 3 + 1\cdot 1\cdot 3)/(2\cdot 3 + 3\cdot 3 + 1\cdot 3) = 0.889$, according to (12).

Step 2) On station $k = N-1 = 2$. Check $\lambda_{2|3}$.

Since $\lambda_{2|3} = 1$, we need not place any sensor on Station II. A numerical calculation conducted by authors verifies that $\mu$ does not change even if additional sensors are placed on Station II.

Step 3) $k = N-2 = 1$. Check $\lambda_{1|3}$, and minimize the number of sensors on Station I to reach $\mu = 1$.

Since $\lambda_{1|3} = 0.667 < 1$, we should place sensors on Station I. Provided that some sensors were already installed on the downstream station, it is

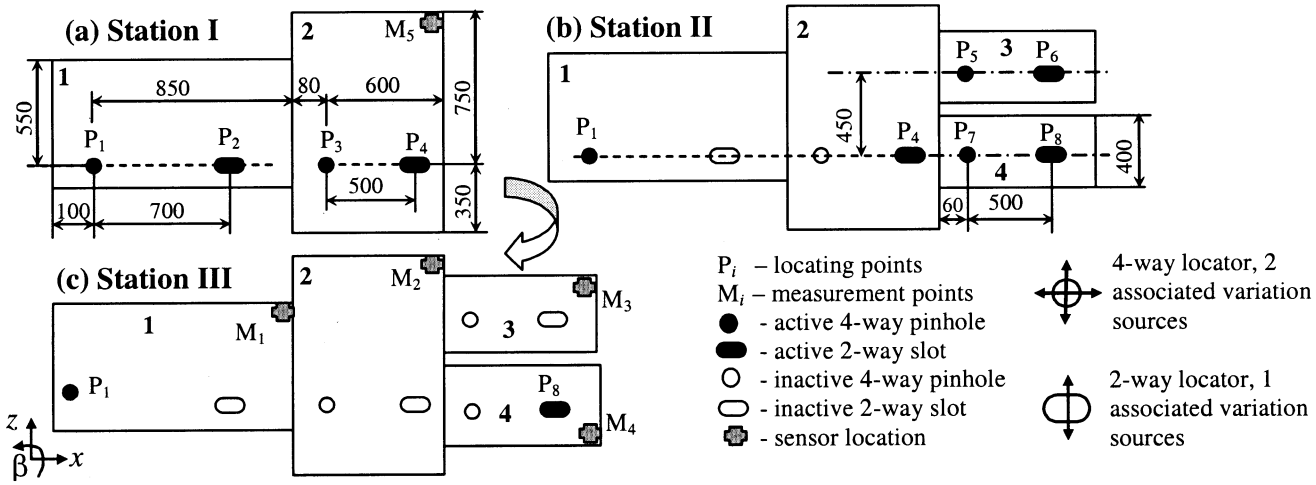


Fig. 5. The three-station assembly process with the optimal sensor distribution.

usually the case that we do not have to install sensors on every part on Station I. A combinatorial test trying different numbers of sensors is necessary to find out the minimum number of sensors required to reach $\mu = 1$. Given $I_k$ parts on station $k$, the maximal number of possible combinations is $\sum_{i=1}^{I_k} C_{I_k}^i$, where $C_a^b$ is the combinatorial operator for integers $a$ and $b$. In this example, $\sum_{i=1}^{I_1} C_{I_1}^i = 3$,

$$\mathbf{A}_1 = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0.0007 & 1 & 0 & -0.0007 & -0.3497 & \\ -1 & 0 & 0 & 1 & 0 & 0 & \mathbf{0}^{6\times6} \\ 0 & -0.3497 & 0 & 0 & 0.3497 & -325.17 & \\ 0 & 0.0007 & 0 & 0 & -0.0007 & 0.6503 & \\ \hline \multicolumn{6}{c|}{\mathbf{0}^{6\times6}} & \mathbf{I}^{6\times6} \end{array}\right]_{12\times12}$$

$$\mathbf{A}_2 = \left[\begin{array}{ccc|c|ccc} 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0}^{3\times6} & 0 & 0 & 0 \\ 0 & 0.0005 & 1 & & 0 & -0.0005 & -0.2392 \\ \hline -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & -0.5550 & 0 & & 0 & -0.4450 & -222.49 \\ 0 & 0.0005 & 0 & \mathbf{I}^{6\times6} & 0 & -0.0005 & -0.2392 \\ -1 & -0.2153 & 0 & & 0 & 0.2153 & 107.655 \\ 0 & -0.2392 & 0 & & 0 & -0.7608 & -380.38 \\ 0 & 0.0005 & 0 & & 0 & -0.0005 & -0.2392 \\ \hline -1 & 0 & 0 & & 1 & -0.0005 & 0 \\ 0 & -0.2392 & 0 & \mathbf{0}^{3\times6} & 0 & 0.2392 & -380.38 \\ 0 & 0.0005 & 0 & & 0 & -0.0005 & 0.7608 \end{array}\right]_{12\times12} \quad (22)$$

$$\mathbf{B}_1 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.0014 & 0.0014 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.002 & 0.002 \\ \hline \multicolumn{6}{c}{\mathbf{0}^{6\times6}} \end{array}\right]_{12\times6}$$

$$\mathbf{B}_2 = \left[\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ 0 & -0.0007 & 0.0007 & & & \mathbf{0}^{6\times6} & & & \\ 1 & 0 & 0 & & & & & & \\ 0 & 0.3497 & 0.6503 & & & & & & \\ 0 & -0.0007 & 0.0007 & & & & & & \\ \hline & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & 0 & -0.002 & 0.002 & 0 & 0 & 0 \\ & \mathbf{0}^{6\times3} & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 0 & -0.002 & 0.002 \end{array}\right]_{12\times9}$$

$$\mathbf{B}_3 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.0005 & 0.0005 \\ 1 & 0 & 0 \\ 0 & 0.5550 & 0.4450 \\ 0 & -0.0005 & 0.0005 \\ 1 & 0.2153 & -0.2153 \\ 0 & 0.2392 & 0.7608 \\ 0 & -0.0005 & 0.0005 \\ 1 & 0 & 0 \\ 0 & 0.2392 & 0.7608 \\ 0 & -0.0005 & 0.0005 \end{array}\right]_{12\times3} \quad (23)$$

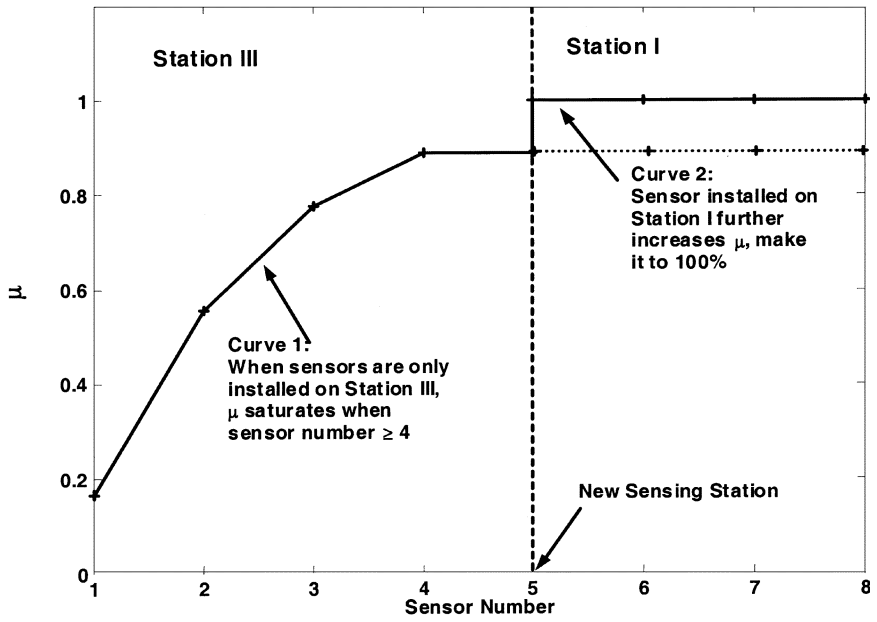


Fig. 6. The impact of sensor number on $\mu$.

where $I_1 = 2$. The three possible sensor placements are (1,0), (0,1), and (1,1). In fact, adding one more sensor on Station I, either on part 1 or on part 2, will result in $\mu = 1$. The optimal sensor distribution is to place a total of five sensors at two stations (marked as $M_1 \sim M_5$ in Fig. 5). In Fig. 5(a), sensor $M_5$ is placed on part 2; but it can alternatively be placed on part 1. Furthermore, we use the algorithm in [14] to determine the exact coordinates of each sensor (see Table V). In order to reduce the computation load, the resolution of coordinates is chosen at the 1-mm level, which should be accurate enough for sensor locations on a part with dimensions of several hundred millimeters.

TABLE V
EXACT COORDINATES OF FIVE SENSORS IN FIG. 5 (UNIT: mm)

| Sensor Points | Coordinates (x, z) |
|---|---|
| $M_1$ (on Station III) | (950, 900) |
| $M_2$ (on Station III) | (1630, 1100) |
| $M_3$ (on Station III) | (2280, 1000) |
| $M_4$ (on Station III) | (2280, 150) |
| $M_5$ (on Station I) | (1630, 1100) |

For such a sensor distribution, matrices $\mathbf{C}$'s are given in (24). Calculate $\mu$ using these $\mathbf{C}$ matrices. Fig. 6 demonstrates that $\mu$ reaches 1 when Station I is upgraded into an additional sensing station, verifying the argument we provided for Fig. 4 in Section III-D, shown in (24) at the bottom of the page.

The abovementioned distributed sensing layout can be compared with two traditional sensing layouts: end-of-line sensing and saturated sensing, which are discussed in [21]. End-of-line sensing layout is defined as placing a sufficient number of sensors at the *last* station to measure the d.o.f of all parts. Saturated sensing layout is defined as placing a sufficient number of sensors to measure the d.o.f of all parts on *every* station. In the cases of end-of-line and saturated sensing layouts, "a sufficient number of sensors" means two sensors per part (for a 2-D assembly process). Thus, in this example, the end-of-line sensing layout will install eight sensors on Station III and the saturated sensing layout needs twenty sensors, two sensors on each part on every station. The results of all three sensing layouts are presented in Table VI.

It shows that the optimal algorithm yields the minimum number of sensors and sensing stations while simultaneously

$$\mathbf{C}_1 = \left[\mathbf{0}^{2\times 3} \left| \begin{matrix} 1 & 0 & -650 \\ 0 & 1 & 550 \end{matrix} \right| \mathbf{0}^{2\times 6}\right]_{2\times 12} \qquad \mathbf{C}_3 = \left[\begin{array}{c|c|c|c} \begin{matrix} 1\ 0\ -450 \\ 0\ 1\ \ 750 \end{matrix} & \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} \\ \hline \mathbf{0}^{2\times 3} & \begin{matrix} 1\ 0\ -650 \\ 0\ 1\ \ 500 \end{matrix} & \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} \\ \hline \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} & \begin{matrix} 1\ 0\ -100 \\ 0\ 1\ \ 490 \end{matrix} & \mathbf{0}^{2\times 3} \\ \hline \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} & \mathbf{0}^{2\times 3} & \begin{matrix} 1\ 0\ 300 \\ 0\ 1\ 490 \end{matrix} \end{array}\right]_{8\times 12} \tag{24}$$

TABLE VI
COMPARISON AMONG SENSOR DISTRIBUTIONS FOR THE THREE-STATION ASSEMBLY

| | # of sensors | # of stations | μ | $J_{opt}$ |
|---|---|---|---|---|
| Saturated sensing | 20 | 3 | 100% | $3c_2+20c_1$ |
| End-of-line sensing | 8 | 1 | 88.9% | $c_2+8c_1$ |
| Optimal strategy | 5 | 2 | 100% | $2c_2+5c_1$ |

attaining 100% system diagnosability ($\mu = 1$). The cost reduction in comparing the optimal sensing strategy with the scheme of saturated sensing is

$$\text{cost reduction in station construction} = \frac{c_2}{3c_2} = 33.3\% \quad (25)$$

$$\text{cost reduction in sensor implementation} = \frac{15c_1}{20c_1} = 75\%. \quad (26)$$

## V. SUMMARY

This paper investigates a strategy for sensor distribution in a multistation assembly system based on the state space variation model and an analytical diagnosability study of the same process. A backward propagation algorithm is presented for the *allocation of sensing stations* along the process and the *determination of the minimal number of sensors within each sensing station*. When the unique properties (Lemmas 1, 2, 3) of variation propagation in a multistation process are considered, the resulting strategy of sensor distribution is optimal, i.e., the sensing cost is minimized. This procedure of sensor distribution is illustrated using one case study of a three-station assembly process. The optimal scheme renders 33% decrease in sensing station cost and 75% decrease in sensor cost in comparison to the scheme of saturated sensing.

Even though the current study focuses on an assembly process and the diagnosis of fixture variation, it is worthwhile to note that the state space model for variation propagation, the diagnosability measure, and the resulting strategy of sensor distribution are fairly general for various types of manufacturing systems. Following the framework, a similar sensor-distribution strategy can be developed for dimensional variation diagnosis in other manufacturing systems.

## APPENDIX I

### PROOF OF (P1) AND (P2) IN LEMMA 1 IN SECTION III-B

*Proof of (P1):* $\lambda_{i|i} = \rho(\pi(\mathbf{\Phi}_{i,i}\mathbf{B}_i))/(p_i \cdot \mathrm{DOF})$. Since $\mathbf{\Phi}_{i,i} = \mathbf{I}$, $\lambda_{i|i} = \rho(\pi(\mathbf{B}_i))/(p_i \cdot \mathrm{DOF})$. Fixture design requires that all degrees of freedom of a rigid workpiece are fully restrained by $p_i$ fixtures on station $i$. Thus, $\rho(\pi(\mathbf{B}_i)) = p_i \cdot \mathrm{DOF}$. That leads to the property (P1). □

*Proof of (P2):* $\lambda_{i|j} = \lambda_{i|k}$ is equivalent to saying $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)) = \rho(\pi(\mathbf{\Phi}_{j,i}\mathbf{B}_i))$. Let us first take a look at $\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i)$. According to [26, Figs. 2–6], $\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i) = \rho(\mathbf{B}_i) - d$, where $d$ is the dimension of the intersection of the column space $R(\mathbf{B}_i)$ and the null space $\mathrm{Null}(\mathbf{\Phi}_{k,i})$.

What is the dimension of $\mathrm{Null}(\mathbf{\Phi}_{k,i})$? The $\mathbf{\Phi}_{k,i}$ is the matrix that models the part transition from station $i$ to station $k$. The part transferring operation happens after all components in an assembly are joined and released from fixtures (Fig. 7). That gives the whole subassembly DOF degrees of freedom before it is restrained in a new fixture on the subsequent station. Therefore, $\mathbf{A}$ has less full rank and so does $\mathbf{\Phi}_{k,i}$ $\forall k \neq i$. The rank of $\mathbf{A}$ should be DOF less than its dimension. Thus, the dimension of $\mathrm{Null}(\mathbf{A})$ is DOF, which is the number of degrees of freedom that the subassembly possessed after being released. This number does not change (it is always DOF) after more parts are assembled into the subassembly, suggesting that the dimension of $\mathrm{Null}(\mathbf{\Phi}_{k,i})$ is also DOF.

However, since the part-positioning described by $\mathbf{B}_i$ takes place at station $i$ before the joining operation (Fig. 7), the deviation among parts can be freely generated by $\mathbf{B}_i$. Therefore, $\mathrm{Null}(\mathbf{\Phi}_{k,i}) \subset R(\mathbf{B}_i)$, suggesting that $d = \mathrm{DOF}$. Given $\rho(\mathbf{B}_i) = p_i \cdot \mathrm{DOF}$, we can have $\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i) = (p_i - 1) \cdot \mathrm{DOF}$. Then, $\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i) = \rho(\mathbf{\Phi}_{j,i}\mathbf{B}_i) = (p_i - 1) \cdot \mathrm{DOF}$, $\forall\, k, j \neq i$.

That is one step short of showing that $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)) = \rho(\pi(\mathbf{\Phi}_{j,i}\mathbf{B}_i))$. Given $\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i) = \rho(\mathbf{\Phi}_{j,i}\mathbf{B}_i)$, if the linear dependent relationship among columns in $\mathbf{\Phi}_{k,i}\mathbf{B}_i$ is the same as that in $\mathbf{\Phi}_{j,i}\mathbf{B}_i$, $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i))$ will equal $\rho(\pi(\mathbf{\Phi}_{j,i}\mathbf{B}_i))$. That is in fact the case. After parts are assembled on station $i$, there forms a certain linear relationship among the columns, e.g., $\sum_{j=1}^{m_i} a_j \boldsymbol{\varphi}_j = \mathbf{0}$, where $a_j$ is a constant, and $\boldsymbol{\varphi}_j$ is the $j$th column vector of $\mathbf{\Phi}_{k,i}\mathbf{B}_i$. This relationship will not change because it is determined by the relative position among all assembled parts in the subassembly, which is fixed by the "joining" operation on station $i$. Thus, after a subassembly is transferred to downstream stations, we can conclude that following relation is true, i.e., $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)) = \rho(\pi(\mathbf{\Phi}_{j,i}\mathbf{B}_i))$ $\forall\, k,\ j \neq i$. Moreover, it was proved in [21] that $\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i)) \geq \rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i)$, thus

$$\lambda_{i|k} = \frac{\rho(\pi(\mathbf{\Phi}_{k,i}\mathbf{B}_i))}{p_i \cdot \mathrm{DOF}} \geq \frac{\rho(\mathbf{\Phi}_{k,i}\mathbf{B}_i)}{p_i \cdot \mathrm{DOF}} = \frac{(p_i - 1) \cdot \mathrm{DOF}}{p_i \cdot \mathrm{DOF}} = \frac{p_i - 1}{p_i}.$$

□

## APPENDIX II

### PROOF OF LEMMA 2 IN SECTION III-B

If sensors are placed on station $N$, $\mathbf{\Gamma}$ in (3) becomes $[\mathbf{\Gamma}_1 \cdots \mathbf{\Gamma}_k \cdots \mathbf{\Gamma}_N]$, where $\mathbf{\Gamma}_k = \mathbf{C}_N\mathbf{\Phi}_{N,k}\mathbf{B}_k$. Meanwhile, given the assumption of "sufficient sensor number," we can set $\mathbf{C}_N = \mathbf{I}$ without loss of generality. Then, $\mathbf{\Gamma}_k$ is further simplified as $\mathbf{\Phi}_{N,k}\mathbf{B}_k$. Given an assembly process where new parts are assembled with an existent subassembly at each intermediate station, the nonzero variables in the state vector $\mathbf{X}$ will keep growing. As a result, the nonzero elements in the columns of $\mathbf{\Gamma}_k$ will increase in accordance to the growth of nonzero elements in $\mathbf{X}_k$. Notice that $I_k$ monotonically increases, i.e., $I_1 < I_2 < \ldots < I_N$, and, $\mathbf{P}_k$ will contribute to deviations only on part 1 to part $I_k$ at station $k$. Hence, $\mathbf{\Gamma}_k$ can be partitioned as

$$\mathbf{\Gamma}_k \equiv \mathbf{\Phi}_{N,k}\mathbf{B}_k = \begin{bmatrix} x \\ \hline \mathbf{0} \end{bmatrix} \begin{matrix} \} \text{ corresponding to part 1 to } I_k \\ \} \text{ corresponding to parts after } I_k+1 \end{matrix} \quad (27)$$

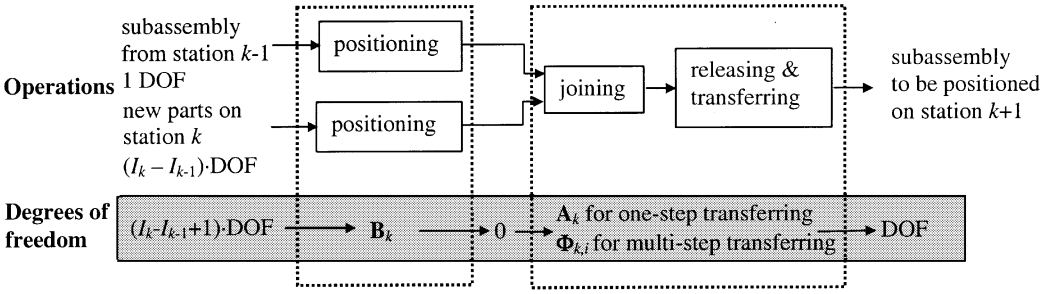


Fig. 7. Assembly operations and their mathematical models.

where "×" is the nonzero block corresponding to part 1 to part $I_k$ that have already been joined together in the assembly after operation on station $k$. Apparently, the columns in $\bar{\mathbf{\Gamma}}_k$ are independent of those in $\mathbf{\Gamma}_j, \forall\ j \neq k$. Based on *Property 2* of $\pi$ transform [21], we know that columns in $\pi(\mathbf{\Gamma}_k)$ will also be independent of those in $\pi(\mathbf{\Gamma}_j)$ for $j \neq k$. Note that $\pi(\mathbf{\Gamma}_k)$ and $\pi(\mathbf{\Gamma}_j)$ are in fact $\mathbf{\Pi}_k$ and $\mathbf{\Pi}_j$ defined in [21], respectively. According to [21, Th. 3], we conclude that the between-station diagnosability condition is satisfied. □

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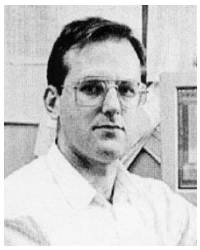

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