A Characterization of Diagnosability Conditions for Variance Components Analysis in Assembly Operations

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***Abstract*—Variance component estimation algorithms, in conjunction with automated in-process measurement technology, can be effective tools for identifying and eliminating major sources of manufacturing variation in assembly processes. Whether a particular set of variation sources are diagnosable depends critically on how the sensor system is laid out. Diagnosability tests are mathematical in nature and provide little insight into why a particular sensor layout may be nondiagnosable or how to modify the layout to ensure diagnosability. This paper translates the mathematical diagnosability conditions into a set of more conceptually meaningful conditions that provide better insight into the reasons behind the nondiagnosability.**

***Note to Practitioners* —This paper was motivated by the problem of identifying and eliminating major sources of variation in discrete-part manufacturing, which are critical steps in improving product quality. The effectiveness of statistical algorithms for estimating sources of variation depends on whether the sensor system for measuring key product and process variables is laid out properly, so that a particular set of diagnosability conditions are satisfied. This paper translates the rather abstract mathematical conditions for diagnosability into a set of more intuitive and conceptually meaningful conditions. This provides practitioners with insight into why a sensor system may be nondiagnosable and how to add or adjust sensors in order to ensure diagnosability. The diagnosability characterization can also be used to enhance performance and reduce computational expense in numerical search strategies for optimizing sensor layout.**

***Index Terms*—Assembly systems, fault diagnosis, manufacturing variation reduction, sensor layout, variance component estimation.**

I. INTRODUCTION

IN RECENT YEARS, there has been considerable work on reducing dimensional variation in assembly processes, in particular in automobile body assembly [1]–[9]. Major advances in this area have occurred in part because of the rapid proliferation of automated in-process dimensional measurement technology, which includes the noncontact laser-optical metrology systems that are now common in automobile manufacturing [1], [7], [8]. The broad objective of the aforementioned work is to effectively utilize the dimensional measurement data for the purpose of identifying (and subsequently eliminating) major root causes of part-to-part dimensional variation. References [3]–[9] have all developed measurement data analysis algorithms for diagnosing root causes of variation, with particular emphasis placed on fixture- and other tooling-related variation sources. In order to facilitate the root cause diagnosis, most of these approaches employ the following linear structured model for representing the effects of the variation sources on the measurement data:

$$\mathbf{y}(t) = \Gamma\mathbf{u}(t) + \mathbf{v}(t) = \sum_{i=1}^{p} \Gamma_i u_i(t) + \mathbf{v}(t), \qquad t = 1, 2, \ldots, N \tag{1}$$

where $\mathbf{y}(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$ is a vector of n measured product features, $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_p(t)]^T$ is a random vector whose elements represent p independent variation sources, $\mathbf{v}(t)$ is an additive random noise vector (e.g., sensor noise), $\Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_p]$ is an $n \times p$ matrix relating the variation sources to the measurement vector, t is an observation index, and N is the sample size. The quantity $\Gamma_i u_i(t)$ therefore represents the effects of the ith variation source on the measurements for part number t of the sample. The sensor system is normally assumed to be homogeneous so that the elements of $\mathbf{v}(t)$ are independent with equal variance σ^2. In other words, the covariance matrix of $\mathbf{v}(t)$ is $\Sigma_v = \sigma^2\mathbf{I}$. Alternative assumptions for the sensor noise are considered in Section V. Because the elements of $\mathbf{u}(t)$ are assumed independent, its covariance matrix is a diagonal matrix $\Sigma_u = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2\}$.

The matrix Γ is assumed available and will depend on a number of factors, including the geometry of the parts, the nature and location of the tooling elements and other variation sources, and the location of the sensors. There have been a number of recent analytical modeling developments for conveniently obtaining Γ based on engineering knowledge of the process physics and a specified set of potential variation sources [10]–[15]. It is also possible to obtain Γ using other means, such as expert knowledge and historical databases of previously identified variation sources.

To illustrate the concepts, consider the following example involving dimensional variability in the liftgate opening of a minivan. More detailed examples and descriptions of the autobody assembly process and the measurement technology can be

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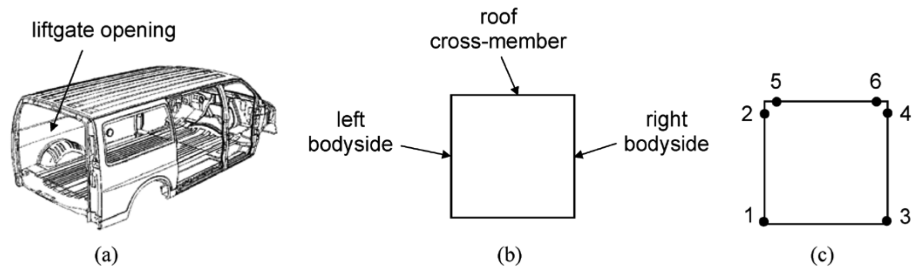

Fig. 1. Illustration of the actual liftgate opening (a), a schematic box representation (b), and a potential sensor layout (c) with six sensors numbered 1 through 6.

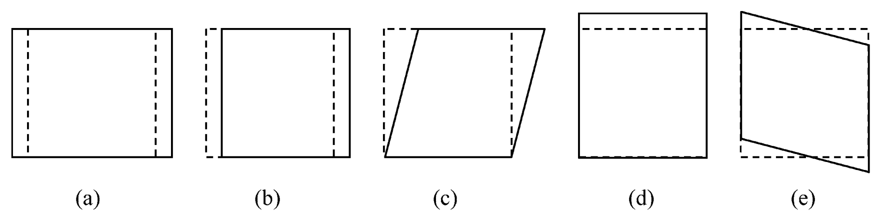

Fig. 2. Illustration of five variation patterns affecting the liftgate opening. (a) Pattern 1—a horizontal enlargement. (b) Pattern 2—a horizontal translation. (c) Pattern 3—a horizontal matchboxing. (d) Pattern 4—a vertical enlargement. (e) Pattern 5—a vertical matchboxing.

found in [2], [8], and [16]. Fig. 1(a) shows the liftgate opening, which is illustrated schematically as a box in Fig. 1(b). Suppose that 6 sensors are positioned around the liftgate opening as in Fig. 1(c). The sensors positioned on the bodyside (sensors 1 through 4) each measure the left/right dimensional deviation from nominal at their particular location. The sensors positioned on the roof cross-member (sensors 5 and 6) each measure the up/down deviation from nominal. Deviations in the up and right directions are taken to be positive. Deviations in the left and down directions are taken to be negative. In this case, N is the number of autobodies in the sample, and t is the autobody index.

Suppose we are interested in diagnosing the five potential variation patterns illustrated in Fig. 2, each of which is a relatively common occurrence as the tooling becomes worn, loose, broken, etc. Throughout, we will refer to the effects of a variation source as a variation pattern and the corresponding column of Γ as a pattern vector. Based on the geometry of the patterns and the locations of the sensors shown in Fig. 1(c), the Γ matrix for this example is

$$\Gamma = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{2}$$

In some cases (e.g., the example in Section IV-A) the elements of $\mathbf{u}$ are associated with distinct physical quantities. In this example, however, the elements of $\mathbf{u}$ are defined implicitly, as the amount of deviation along each of the patterns shown in Fig. 2. For example, $u_1(t)$ represents the amount the liftgate opening enlarges on autobody number $t, u_2(t)$ represents the amount the liftgate opening translates on autobody number t, etc. Note that each pattern represents part-to-part variation, as opposed to a mean shift. For example, although pattern 1 is shown as a positive enlargement in Fig. 2(a), on some autobodies in the sample the enlargement may be negative (a contraction) depending on whether the value of $u_1(t)$ was positive or negative for that autobody.

The diagnostic objective in this paper and in most of the aforementioned references is to estimate the variance components $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2\}$ for each of the p potential variation sources, based on the data sample $\{\mathbf{y}(t) : t = 1, 2, \ldots, N\}$. Knowledge of the variance components allows us to assess whether each variation source is present in the current sample and, if it is, the severity of the source. All of the existing diagnostic algorithms (the most effective of which are summarized in Section II) require a set of diagnosability conditions to be satisfied in order to produce valid, unique estimates of the variance components. This is analogous to the issue of singularity in standard least squares, which results in nonunique parameter estimates. Although there are straightforward mathematical tests of diagnosability that typically involve checking whether a certain matrix is singular [17], [18], they provide little insight into why a system is nondiagnosable or how to add sensors or adjust their locations to ensure diagnosability. The main purpose of this paper is to translate the mathematical conditions for diagnosability into a set of more conceptually meaningful conditions that provides better insight into the reasons behind the diagnosability problems.

The format of the remainder of the paper is as follows. In Section II, we review algorithms for estimating the variance components, as well as their mathematical diagnosability conditions. In Section III, we characterize the diagnosability conditions in terms of a set of more conceptually meaningful conditions. Section IV provides some interpretations of the conditions and illustrates with examples. In Section V, we discuss how the results extend to variants of the algorithm developed under different assumptions on the sensor noise. Section VI concludes the paper.

II. Review of Variance Components Estimation Algorithm

Throughout, the "$\wedge$" overscore symbol denotes an estimate of a quantity. Unless otherwise noted, we assume the sample mean $\bar{\mathbf{y}} = N^{-1}\sum_{t=1}^{N}\mathbf{y}(t)$ has been subtracted from the data so that the resulting sample $\{\mathbf{y}(t) : t = 1, 2, \ldots, N\}$ can be taken to be zero-mean. The motivation for the primary algorithm considered in this paper (refer to [19] for details) is as follows. Express the covariance matrix of $\mathbf{y}$ as $\Sigma_y = \sum_{i=1}^{p}\Gamma_i\Gamma_i^T\sigma_i^2 + \sigma^2\mathbf{I}$, and consider the sample covariance matrix $\mathbf{S}_y = (N-1)^{-1}\sum_{t=1}^{N}\mathbf{y}(t)\mathbf{y}^T(t)$ as an estimate of Σ_y. The variance component estimates are taken to be the values that minimize the sum of the squares of the elements of the error matrix $\mathbf{S}_y - \sum_{i=1}^{p}\Gamma_i\Gamma_i^T\hat{\sigma}_i^2 - \hat{\sigma}^2\mathbf{I}$. The estimates for this approach, which we refer to as matrix least squares (MLS), are given by

$$\hat{\boldsymbol{\sigma}} = \mathbf{G}^{-1}\mathbf{b} \tag{3}$$

where $\hat{\boldsymbol{\sigma}} = [\hat{\sigma}_1^2, \hat{\sigma}_2^2, \ldots, \hat{\sigma}_p^2, \hat{\sigma}^2]^T$ is the vector of variance component estimates

$$\mathbf{G} = \begin{bmatrix} \left(\Gamma_1^T\Gamma_1\right)^2 & \cdots & \left(\Gamma_1^T\Gamma_p\right)^2 & \Gamma_1^T\Gamma_1 \\ \vdots & & & \vdots \\ \left(\Gamma_1^T\Gamma_p\right)^2 & \cdots & \left(\Gamma_p^T\Gamma_p\right)^2 & \Gamma_p^T\Gamma_p \\ \Gamma_1^T\Gamma_1 & \cdots & \Gamma_p^T\Gamma_p & n \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} \Gamma_1^T\mathbf{S}_y\Gamma_1 \\ \vdots \\ \Gamma_p^T\mathbf{S}_y\Gamma_p \\ \mathrm{tr}(\mathbf{S}_y) \end{bmatrix}. \tag{4}$$

The diagnosability condition for the MLS algorithm is that $\mathbf{G}$ has full rank $p+1$, so that its inverse exists in (3). Because $\mathbf{G}$ is a Gram matrix of (appropriately defined) matrix inner products between pairs of matrices in the set $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_p\Gamma_p^T, \mathbf{I}\}$, an equivalent condition for diagnosability is that these matrices are linearly independent. Otherwise, the variance component estimates that minimize the MLS error criterion are not unique.

It should be noted that the variance components estimation algorithms discussed in this paper assume that all variation sources that are present in the data are included in the model. If a variation source is present in the data but is not included in the Γ matrix, then that will introduce bias in the estimates of the other variance components. Although the missing variation source could be considered part of the noise vector, this would result in correlated noise with unequal variances, which violates other assumptions of the methods.

It should also be noted that variance components estimation algorithms are intended to diagnose sources of *variation*, as opposed to *mean shifts*. Any mean shift that is constant over the data sample will have no effect on the variance components estimation, because the sample average $\bar{\mathbf{y}}$ is subtracted from the data when calculating the sample covariance matrix. Although mean shifts often have large impact on dimensional integrity, entirely different methods are required in order to diagnose them because of their constant nature.

A variant of the MLS algorithm was developed in [20] as an approximate maximum likelihood method and also in [19] as a weighted least squares version of the MLS algorithm that was designed to improve estimation accuracy. This weighted MLS algorithm is an iterative procedure that involves pre- and post-multiplying the error matrix $\mathbf{S}_y - \sum_{i=1}^{p}\Gamma_i\Gamma_i^T\hat{\sigma}_i^2 - \hat{\sigma}^2\mathbf{I}$ in the MLS criterion by an appropriately chosen weighting matrix that depends on the current estimates of the variance components at each iteration (refer to [19] for details). Because the focus of this paper is diagnosability, and the diagnosability conditions for the weighted and unweighted MLS algorithms are equivalent [19], we will subsequently make no distinction between these two algorithms and refer to both as simply the MLS algorithm.

One might consider using a more conventional alternative to the MLS algorithm, such as the standard least squares (LS) algorithm discussed in [4]. In this approach, one calculates the LS estimate $\hat{\mathbf{u}}(t) = [\Gamma^T\Gamma]^{-1}\Gamma^T\mathbf{y}(t)$ for each $t = 1, 2, \ldots, N$. The variance components are then estimated using the sample variances of the elements of $\hat{\mathbf{u}}(\cdot)$. This approach clearly requires that $\Gamma^T\Gamma$ is invertible or, equivalently, that the pattern vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ are linearly independent. The exact diagnosability conditions for the LS algorithm are that $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ are linearly independent and that $n > p$ (the strict inequality results from the manner in which the noise variance is estimated [4], [19]). Reference [19] demonstrated that the MLS algorithm generally outperforms the LS algorithm. Consequently, this paper focuses primarily on the MLS algorithm.

Reference [19] also showed that diagnosability of the LS algorithm always implies diagnosability of the MLS algorithm, but that the converse is not necessarily true. In particular, there are situations in which the pattern vectors are linearly dependent, but the MLS algorithm remains diagnosable. In fact, the MLS algorithm may be diagnosable even in situations in which n is substantially less than p. Although this may violate intuition, there is a reasonable explanation. In the MLS algorithm, we are not attempting to estimate the elements of $\mathbf{u}(t)$ directly. We are only interested in estimating their *variances*, and for this it is not necessary that the columns of Γ are linearly independent. In contrast, because the LS algorithm does attempt to estimate the elements of $\mathbf{u}(t)$ directly, it involves the stricter requirement that Γ have linearly independent columns. In particular, it requires that we have at least as many sensors as we have variation sources.

References [19] and [18] provided examples in which the pattern vectors were linearly dependent but the MLS algorithm was still diagnosable. However, the reasons why certain types of linear dependencies are allowable, whereas others are not, are not well understood. The singularity of the Gram matrix $\mathbf{G}$ is easy to check mathematically, but it is difficult to interpret and provides little insight into why a system is or is not diagnosable. The remainder of this paper attempts to provide insight into this issue by characterizing the type of linear dependencies that result in a nondiagnosable MLS algorithm, versus the type of linear dependencies that still allow diagnosability.

III. Characterization of Diagnosability Conditions

In this section, we present theorems that translate diagnosability (nonsingularity of the Gram matrix $\mathbf{G}$) into conditions on the pattern vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$. We first present a necessary and sufficient set of conditions for the MLS algorithm to be nondiagnosable for the case that $n > p$. Results for the case where $n \leq p$ follow.

Theorem 1: For $n > p$, a necessary and sufficient condition for the MLS algorithm to be nondiagnosable is that all of the following hold.

C1) There must exist a set of $l \geq 1$ pattern vectors, each of which can be written as a linear combination of a second disjoint set of k pattern vectors.

C2) $k \leq l$.

C3) Denote the two sets of pattern vectors in condition (C1) as $\bar{\Gamma} = [\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_l}]$ and $\underline{\Gamma} = [\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_k}]$ with $\{i_1, i_2, \ldots i_l\} \subset \{1, 2, \ldots, p\}$ and $\{j_1, j_2, \ldots j_k\} \subset \{1, 2, \ldots, p\}$ two disjoint sets of indices. Write their relationship as $\bar{\Gamma} = \underline{\Gamma}\boldsymbol{\beta}$ for some $k \times l$ matrix $\boldsymbol{\beta}$. There must exist a nonzero diagonal matrix $\bar{\mathbf{A}} = \mathrm{diag}\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{il}\}$ such that $\boldsymbol{\beta}\bar{\mathbf{A}}\boldsymbol{\beta}^T$ is diagonal.

Proof: We first prove necessity. If the MLS algorithm is nondiagnosable, then the set of matrices $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_p\Gamma_p^T, \mathbf{I}\}$ are linearly dependent, in which case there exists a set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_p, \delta\}$, not all zero, such that $\alpha_1\Gamma_1\Gamma_1^T + \alpha_2\Gamma_2\Gamma_2^T + \cdots + \alpha_p\Gamma_p\Gamma_p^T = -\delta\mathbf{I}$. In order for this to hold, we must have $\delta = 0$. Otherwise, $\mathrm{rank}(\delta\mathbf{I}) = n$, whereas the summation of matrices on the left hand side can have at most rank $p < n$. Let m denote the number of nonzero α_i's and order the pattern vectors so that $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ are nonzero and $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\}$ are linearly independent, where $k = \mathrm{rank}[\Gamma_1, \Gamma_2, \ldots, \Gamma_m]$. Define $\underline{\Gamma} = [\Gamma_1, \Gamma_2, \ldots, \Gamma_k], \bar{\Gamma} = [\Gamma_{k+1}, \Gamma_{k+2}, \ldots, \Gamma_m], \underline{\mathbf{A}} = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, and $\bar{\mathbf{A}} = \mathrm{diag}\{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_m\}$. Because $\mathrm{rank}[\Gamma_1, \Gamma_2, \ldots, \Gamma_m] = k$, each column of $\bar{\Gamma}$ must be a linear combination of the columns of $\underline{\Gamma}$. Write this as $\bar{\Gamma} = \underline{\Gamma}\boldsymbol{\beta}$ for some $k \times l$ matrix $\boldsymbol{\beta}$, where $l = m - k$. We must have $l \geq 1$, because if $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_m\Gamma_m^T\}$ are linearly dependent, then so must be $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$. This proves the necessity of condition (C1).

To prove the necessity of condition (C3), rewrite the relationship $\alpha_1\Gamma_1\Gamma_1^T + \alpha_2\Gamma_2\Gamma_2^T + \cdots + \alpha_m\Gamma_m\Gamma_m^T = 0$ as $\underline{\Gamma}\,\underline{\mathbf{A}}\,\underline{\Gamma}^T = -\bar{\Gamma}\bar{\mathrm{A}}\bar{\Gamma}^T = -\underline{\Gamma}\boldsymbol{\beta}\bar{\mathrm{A}}\boldsymbol{\beta}^T\underline{\Gamma}^T$. Because the k columns of $\underline{\Gamma}$ are linearly independent, this implies that $\boldsymbol{\beta}\bar{\mathrm{A}}\boldsymbol{\beta}^T = -\underline{\mathbf{A}}$, which proves the necessity of condition (C3). From this last equality, it also follows that $k = \mathrm{rank}(\underline{\mathbf{A}}) = \mathrm{rank}(\boldsymbol{\beta}\bar{\mathrm{A}}\boldsymbol{\beta}^T) \leq \mathrm{rank}(\bar{\mathrm{A}}) = l$, which proves the necessity of condition (C2). It is straightforward to show that conditions (C1) through (C3) together imply that $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_p\Gamma_p^T\}$ are linearly dependent, which proves the sufficiency part of the theorem. ■

Remark 1: Condition (C1) holds iff the full set of pattern vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ are linearly dependent. Thus, for $n > p$, condition (C1) alone is a necessary and sufficient condition for the LS algorithm to be nondiagnosable. In addition to this, conditions (C2) and (C3) must also hold in order for the MLS algorithm to be nondiagnosable. Clearly, the MLS algorithm is diagnosable for a broader class of problems. Examples where the MLS algorithm is diagnosable, but the LS algorithm is not, are given later.

The following theorem provides similar results for the case that $n = p$.

Theorem 2: For $n = p$, a necessary and sufficient condition for the MLS algorithm to be nondiagnosable is that *either* condition (C4) *or* condition (C5) below holds.

C4) All of conditions (C1) through (C3) hold.

C5) The $n = p$ columns of Γ are orthogonal.

Proof: We first prove necessity. As in the proof of Theorem 1, if the MLS algorithm is nondiagnosable, then there exists a set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_p, \delta\}$, not all zero, such that $\alpha_1\Gamma_1\Gamma_1^T + \alpha_2\Gamma_2\Gamma_2^T + \cdots + \alpha_p\Gamma_p\Gamma_p^T = -\delta\mathbf{I}$. If $\delta = 0$, then the necessity of condition (C4) is proven as in Theorem 1. If $\delta \neq 0$, write the linear dependency as $\Gamma\mathbf{A}\Gamma^T = -\delta\mathbf{I}$, where $\mathbf{A} = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$. It follows that $n = \mathrm{rank}(\delta\mathbf{I}) = \mathrm{rank}(\Gamma\mathbf{A}\Gamma^T) \leq \mathrm{rank}(\Gamma)$, so that the $n \times n$ matrix Γ has full rank n and is invertible. Thus, $\Gamma^{-1}\Gamma\mathbf{A}\Gamma^T\Gamma^{-T} = -\delta\Gamma^{-1}\mathbf{I}\Gamma^{-T}$, or $\mathbf{A} = -\delta[\Gamma^T\Gamma]^{-1}$. This implies that $[\Gamma^T\Gamma]^{-1}$ is diagonal, which requires that the full-rank $\Gamma^T\Gamma$ is diagonal. In other words, the columns of Γ are orthogonal, so that (C5) is a necessary condition if $\delta \neq 0$.

The sufficiency of condition (C4) can be proven as in Theorem 1. To prove the sufficiency of condition (C5), suppose the columns of Γ are orthogonal. It follows that $\Gamma^T\Gamma = \mathbf{A}$ with $\mathbf{A} = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and each $\alpha_i = \Gamma_i^T\Gamma_i > 0$. Pre- and post-multiplying by $\mathbf{A}^{-1/2} = \mathrm{diag}\{\alpha_1^{-1/2}, \alpha_2^{-1/2}, \ldots, \alpha_p^{-1/2}\}$ gives $\mathbf{A}^{-1/2}\Gamma^T\Gamma\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}\mathbf{A}^{-1/2} = \mathbf{I}$. The matrix $\Gamma\mathbf{A}^{-1/2}$ is therefore orthogonal, so that $\mathbf{I} = \Gamma\mathbf{A}^{-1/2}[\Gamma\mathbf{A}^{-1/2}]^T = \Gamma\mathbf{A}^{-1}\Gamma^T = \sum_{i=1}^{p} \alpha_i^{-1}\Gamma_i\Gamma_i^T$. This implies that the set $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_p\Gamma_p^T, \mathbf{I}\}$ is linearly dependent, in which case the MLS algorithm is nondiagnosable. ■

For $n < p$, the full set of necessary conditions for nondiagnosability is more involved than when $n \geq p$. However, a straightforward repetition of the proofs of sufficiency in Theorems 1 and 2 yields a similar set of sufficient conditions stated in Theorem 3 below. Note that the sample covariance matrix $\mathbf{S}_y$ is symmetric and contains only $n(n+1)/2$ nonredundant elements, which must be used to estimate $p+1$ unknown quantities in the MLS algorithm. Consequently, the absolute minimum number of sensors required for diagnosability must satisfy $n(n+1)/2 \geq p+1$.

Theorem 3: For $n < p$, a sufficient condition for the MLS algorithm to be nondiagnosable is that *either* condition (C4) *or* condition (C6) below holds.

C6) Γ contains n orthogonal columns.

Remark 2: Regardless of n and p, conditions (C1) through (C3) are always sufficient conditions for the MLS algorithm to be nondiagnosable. Although the meaning of conditions (C1) and (C2) are quite straightforward, the meaning of condition (C3) is less intuitively clear. Fortunately, we can often bypass

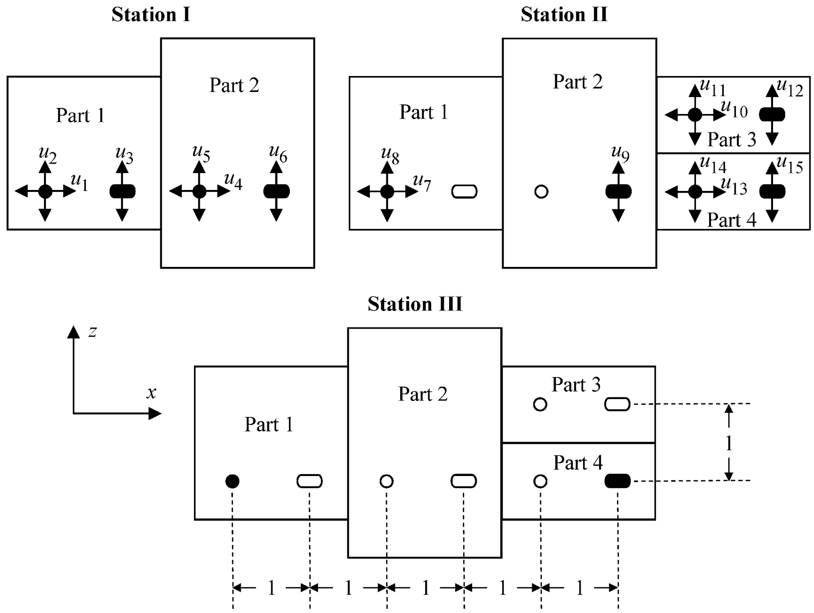

Fig. 3. Three-station assembly system where Parts 1 and 2 are joined in Station I; the Part 1–2 subassembly is joined to Parts 3 and 4 in Station II; and the final assembly is measured in Station III.

this condition. When conditions (C1) and (C2) hold with $k \leq 2$, condition (C3) can be neglected because it will hold automatically. Indeed, if $k = 1$, then $\boldsymbol{\beta}\bar{\mathbf{A}}\boldsymbol{\beta}^T$ is a scalar and condition (C3) holds trivially. If $k = 2, \boldsymbol{\beta}$ is a $2 \times l$ matrix. It is straightforward to verify that for any $2 \times l$ $\boldsymbol{\beta}$ with $l \geq 2$, we can always find a nontrivial diagonal $\bar{\mathbf{A}}$ so that $\boldsymbol{\beta}\bar{\mathbf{A}}\boldsymbol{\beta}^T$ is diagonal. More generally, it can be shown that condition (C3) holds automatically whenever conditions (C1) and (C2) hold with $l \geq 0.5k(k-1)+1$.

IV. Discussion and Examples of Diagnosability Conditions

The results in the previous section translate the singularity of the Gram matrix $\mathbf{G}$ into a set of conditions on the pattern vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$. The pattern vectors are generally much easier to interpret and to work with than the Gram matrix. Engineers usually have a clearer conceptual understanding of how a modification to the sensor layout will affect the pattern vectors than of how it will affect the Gram matrix. This section discusses various implications of the characterization of diagnosability in terms of the pattern vectors and provides illustrative examples.

A. Diagnosable Versus Nondiagnosable Linear Dependencies

One conclusion of the previous section was that there are certain types of linear dependencies among the pattern vectors that result in nondiagnosability, and other types of linear dependencies that are allowable. In light of Remark 2, we can focus primarily on conditions (C1) and (C2) to understand what types of linear dependencies are nondiagnosable. For nondiagnosable linear dependencies, there must exist a set of l pattern vectors $\{\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_l}\}$ that can be written as a linear combination of a set $\{\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_k}\}$ of $k \leq l$ other pattern vectors. We can view this as being able to "cancel out" the effects of each of $\{\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_l}\}$ with a single set $\{\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_k}\}$ of pattern vectors that are of equal of fewer number. The simplest example of this is when $k = l = 1$, in which case we have two collinear pattern vectors. This is obviously a sufficient condition for nondiagnosability, because there is no way to distinguish between the effects of two collinear pattern vectors.

The situation when $k = 2$ is less obvious. If we have only a single pattern vector (say Γ_3) that can be written as a linear combination of $k = 2$ other pattern vectors (say Γ_1 and Γ_2), then the system may still be diagnosable. On the other hand, if we can also write a second pattern vector (say Γ_4) as a linear combination of the same Γ_1 and Γ_2, then the system will always be nondiagnosable. For $k = 3$, if we have $l \geq 0.5k(k-1)+1 = 4$ pattern vectors that can be written as a linear combination of three other pattern vectors, then the system will always be nondiagnosable. This is illustrated in the following example from rigid panel assembly, which is similar to the example considered in [18].

The assembly process shown in Fig. 3 welds four parts together in two stations (Stations I and II). In Station I, Parts 1 and 2 are joined, and the resulting subassembly is joined to Parts 3 and 4 in Station II. In an assembly station, each part (or subassembly) is located in a fixture using a pin that mates with a hole in the part and second pin that mates with a slot in the part. A pin/hole combination constrains two degrees-of-freedom and a pin/slot constrains only one degree-of-freedom. Together, a pin/hole and pin/slot completely constrain all three degrees-of-freedom of the part in the x-z plane. The active holes and slots at each station are shown in solid dark color. Holes and slots that are not shown darkened are not used in that particular station. Station III is a measurement station in which no assembly takes place (measurement is not restricted to Station III, however). The distances between the holes and slots are shown in

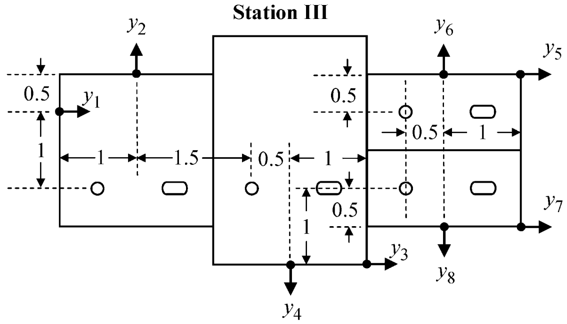

Fig. 4. Nondiagnosable sensor layout for the assembly system shown in Fig. 3.

the Station III figure (the units have no physical meaning and have been scaled for convenience).

The variation sources we will consider are deviations of the pin/hole combinations in the x- and z-directions and deviations of the pin/slot combinations in the z-direction for the two assembly stations. Because there are a total of five active pin/hole combinations and five active pin/slot combinations in Stations I and II, there are a total of 15 variation sources. These are indicated by the arrows labeled u_1 through u_{15} in the Station I and Station II figures. Deviations in the positive x- and positive z-directions are taken to be positive.

Suppose we are considering a sensor layout in which we place two sensors (one measuring the x-deviation and one measuring the z-deviation) on each of the four parts in Station III. One such layout is shown in Fig. 4. The locations of the eight sensors are labeled y_1 through y_8, where the direction of the arrows indicate whether the x-coordinate or z-coordinate is being measured. Based on the geometry of the parts and fixtures shown in Fig. 3 and the positions of the sensors shown in Fig. 4, we can calculate Γ as shown in the equation at the bottom of the page (the bar overscore indicates a repeating digit). The reader is referred to [7], [10], or [12], for a systematic procedure for modeling the Γ matrix in multi-station assembly processes such as this. In this particular example, the geometry was simple enough to obtain Γ using basic kinematic arguments. Note that the above expression for Γ reflects the fact that deviations at one station may be partially compensated at downstream stations. For example, a u_6 deviation in Station I causes Part 2 to rotate about the u_4/u_5 pin. However, because the u_6 slot is an active locator in Station II (in which it is constrained to mate with the u_9 pin), the Part 1/Part 2 subassembly will rotate in the opposite direction to some extent when it is placed in the Station II fixture.

It can be verified that the resulting Gram matrix is singular, so that the system is nondiagnosable. Because we are considering 15 pattern vectors but using only 8 sensors, there are obviously a number of linear dependencies among the columns of Γ. The row-reduced echelon form [21] of Γ was used to identify the following:

$$\begin{aligned}
\Gamma_4 &= -\Gamma_1 \\
\Gamma_5 &= -3\Gamma_2 - 2\Gamma_3 \\
\Gamma_6 &= 2\Gamma_2 + \Gamma_3 \\
\Gamma_9 &= -2\Gamma_1 - 5\Gamma_2 - 3\Gamma_3 \\
\Gamma_{12} &= -\Gamma_{10} + \Gamma_{11} \\
\Gamma_{13} &= -\Gamma_7 - \Gamma_{10} \\
\Gamma_{14} &= 4\Gamma_1 + 10\Gamma_2 + 6\Gamma_3 - 5\Gamma_8 - \Gamma_{10} - \Gamma_{11} \\
\Gamma_{15} &= -2\Gamma_1 - 5\Gamma_2 - 3\Gamma_3 + 4\Gamma_8 + 2\Gamma_{10} - \Gamma_{11}.
\end{aligned}$$

Not all of the linear dependencies are responsible for the nondiagnosability, however. The only linear dependencies that violate conditions (C1) and (C2) are $\Gamma_4 = -\Gamma_1$ [$k = l = 1$ in condition (C2)], and Γ_5 and Γ_6 both being a linear combination of Γ_2 and Γ_3 [$k = l = 2$ in condition (C2)]. The other linear dependencies do not contribute to the nondiagnosability. To substantiate this claim, suppose hypothetically that variation sources 4 and 5 did not exist and we were only interested in estimating the remaining 13 variation sources. If we eliminate the fourth and fifth columns of the above Γ and consider the resulting 8×13 matrix as the new Γ, then we still have all of the above linear dependencies except the first two. The remaining six linear dependencies do not violate conditions (C1) and (C2), however. Consequently, even though we would still have a number of linear dependencies among the columns of Γ, the system would be diagnosable.

Because we want to include variation sources 4 and 5 in the diagnosis, the only recourse is to modify the sensor layout. Knowledge of the offending linear dependencies can be useful for this purpose. Recall that the offending linear dependencies are i) Γ_4 is collinear with Γ_1, and ii) Γ_5 and Γ_6 are both linear combinations Γ_2 and Γ_3. In order to avoid this in the modified sensor layout, we must place a sensor at Station I. If we move the y_1 sensor from Station III to the exact same position on Part 1 but in Station I, the new Γ matrix becomes (only the first row changes) as shown in (5) at the bottom of the next page. It can be verified that the Gram matrix for this Γ is full rank, so that the new sensor layout is diagnosable. This is in spite of the fact that

$$\Gamma = \begin{bmatrix}
0 & 0.\bar{6} & -1 & 0 & 0 & 0.\bar{3} & 0 & 0.1\bar{3} & -0.\bar{3} & 0 & 0 & 0 & 0 & 0 & 0.2 \\
0 & -0.\bar{3} & 0.5 & 0 & 0 & -0.1\bar{6} & 0 & -.0\bar{6} & 0.1\bar{6} & 0 & 0 & 0 & 0 & 0 & -0.1 \\
-1 & 0.\bar{3} & 0 & 1 & -1 & 0.\bar{6} & 0 & -0.1\bar{3} & 0.\bar{3} & 0 & 0 & 0 & 0 & 0 & -0.2 \\
0 & -0.1\bar{6} & 0 & 0 & 0.5 & -0.\bar{3} & 0 & -0.\bar{3} & 0.8\bar{3} & 0 & 0 & 0 & 0 & 0 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -0.3 & 0 & 1 & 0.5 & -0.5 & 0 & 0 & 0.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & -0.9 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.1 & 0 & 0 & 0 & 0 & 1 & -0.5 & 0.4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.4
\end{bmatrix}$$

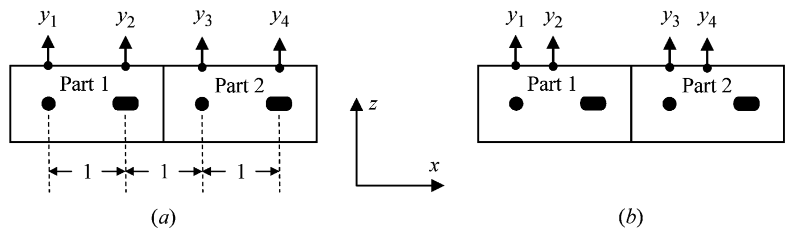

Fig. 5. Two-part assembly with four sensors and two different sensor layouts. (a) Four variation patterns are decoupled and nondiagnosable. (b) Four variation patterns are diagnosable.

we are estimating 15 variance components with only 8 sensors and still have the following linear dependencies:

$$\begin{aligned}\Gamma_5 &= \Gamma_1 - 3\Gamma_2 - 2\Gamma_3 + \Gamma_4\\ \Gamma_6 &= -\Gamma_1 + 2\Gamma_2 + \Gamma_3 - \Gamma_4\\ \Gamma_9 &= 2\Gamma_1 - 5\Gamma_2 - 3\Gamma_3 + 4\Gamma_4\\ \Gamma_{12} &= -\Gamma_{10} + \Gamma_{11}\\ \Gamma_{13} &= -\Gamma_7 - \Gamma_{10}\\ \Gamma_{14} &= -4\Gamma_1 + 10\Gamma_2 + 6\Gamma_3 - 8\Gamma_4\\ &\quad - 5\Gamma_8 - \Gamma_{10} - \Gamma_{11}\\ \Gamma_{15} &= 2\Gamma_1 - 5\Gamma_2 - 3\Gamma_3 + 4\Gamma_4\\ &\quad + 4\Gamma_8 + 2\Gamma_{10} - \Gamma_{11}.\end{aligned}$$

The reason these linear dependencies do not cause nondiagnosability is that they do not violate conditions (C1) and (C2).

B. Implications of the Orthogonality Conditions

The orthogonality condition (C6) also has a straightforward interpretation. An alternative proof that it is a sufficient condition for nondiagnosability uses the following simple argument. Suppose we have a set of n orthogonal pattern vectors (say Γ_1 through Γ_n) with $p \geq n$. Define the $n \times n$ orthogonal matrix $\underline{\Gamma} = [\Gamma_1, \Gamma_2, \ldots, \Gamma_n]$ and the diagonal matrix $\underline{\mathbf{A}} = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$. The covariance matrix of $\mathbf{y}$ becomes

$$\begin{aligned}\Sigma_y &= \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \sigma_i^2 + \sigma^2 \mathbf{I}\\ &= \underline{\Gamma}\ \underline{\mathrm{A}}\ \underline{\Gamma}^T + \sigma^2\mathbf{I} + \sum_{i=n+1}^{p} \Gamma_i \Gamma_i^T \sigma_i^2\\ &= \underline{\Gamma}[\underline{\mathrm{A}} + \sigma^2 \mathbf{I}]\underline{\Gamma}^T + \sum_{i=n+1}^{p} \Gamma_i \Gamma_i^T \sigma_i^2.\end{aligned}$$

If we add any constant to the variance components $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$ and subtract the same constant from σ^2, then we clearly do not change the covariance of $\mathbf{y}$. The variance components estimates are therefore not unique, and the system is nondiagnosable. Note that this is not problematic if we have an *a priori* estimate of the noise variance, as will be discussed in Section V.

The orthogonality condition has implications when laying out a sensor system. One implication is that we should avoid "decoupling" the variance components, as illustrated in the following example. Fig. 5(a) shows an assembly station in which two parts are joined. Suppose we are not concerned with the x-direction displacement of the parts and are only considering the four variance components that represent the z-direction deviations of the four pin/hole or pin/slot combinations. Because the four sensors in layout (a) are located directly above the four pins, the layout decouples the effects of the four variation patterns (each pattern affects a single distinct sensor). The resulting Γ matrix is the 4×4 identity matrix, and the sensor layout in Fig. 5(a) is therefore nondiagnosable. If we move sensors 2 and 4 to positions that are half way between the pins, as shown in Fig. 5(b), the variation patterns are no longer decoupled. Pattern 1 now affects both y_1 and y_2, and pattern 3 affects both y_3 and y_4. In this case

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0.5 & 0.5 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

no longer has orthogonal columns. It can be verified that the Gram matrix is full rank, and the system is diagnosable.

Although the nondiagnosability of the layout in Fig. 5(a) may seem counterintuitive, it is because of the presence of noise. If there were no noise ($\sigma^2 = 0$), or more generally if σ^2 were known, then the layout in Fig. 5(a) would indeed be diagnosable. This is implied by Corollary 1 of Section V.

$$\Gamma = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -0.\bar{3} & 0.5 & 0 & 0 & -0.1\bar{6} & 0 & -.0\bar{6} & 0.1\bar{6} & 0 & 0 & 0 & 0 & 0 & -0.1\\ -1 & 0.\bar{3} & 0 & 1 & -1 & 0.\bar{6} & 0 & -0.1\bar{3} & 0.\bar{3} & 0 & 0 & 0 & 0 & 0 & -0.2\\ 0 & -0.1\bar{6} & 0 & 0 & 0.5 & -0.\bar{3} & 0 & -0.\bar{3} & 0.8\bar{3} & 0 & 0 & 0 & 0 & 0 & -0.5\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -0.3 & 0 & 1 & 0.5 & -0.5 & 0 & 0 & 0.3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & -0.9\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.1 & 0 & 0 & 0 & 0 & 1 & -0.5 & 0.4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.4 \end{bmatrix}. \tag{5}$$

The orthogonality condition also comes into play in the liftgate opening example discussed in Section I with the five variation patterns shown in Fig. 2. Suppose that we can place sensors anywhere on the roof cross-member (measuring up/down deviation from nominal) and/or on the left and right bodysides (measuring left/right deviation). We would like to use the smallest number of sensors that will still result in diagnosability for the five variation sources. Because patterns 4 and 5 affect only roof measurements (and not the bodysides), we need at least two sensors on the roof. Otherwise, if we had only one sensor on the roof, Γ_4 and Γ_5 would be collinear and violate conditions (C1) and (C2) with $k = l = 1$. Similarly, because patterns 1 through 3 affect only the bodysides, we need at least two sensors on the bodysides. Otherwise, if we had only one sensor on the bodysides, Γ_1, Γ_2, and Γ_3 would all be collinear and violate conditions (C1) and (C2) with $k = 1$ and $l = 2$. Consequently, we need at least 4 sensors—2 on the roof and 2 on the bodysides.

In order to determine a suitable layout for the two bodyside sensors, we can again consider the collinearity condition. The two sensors cannot be located on the same bodyside [for example, sensors 1 and 2 in Fig. 1(c)]. Otherwise, Γ_1 and Γ_2 will be collinear. They cannot be located at the same height on opposite bodysides [for example, sensors 2 and 4 in Fig. 1(c)]. Otherwise, Γ_2 and Γ_3 will be collinear. In light of these constraints, we must place the 2 bodyside sensors on opposite sides and at different heights. For example, we could place the sensors at locations 2 and 3 in Fig. 1(c) or at locations 1 and 4.

In order to determine where to place the roof sensors, consider that if we place the bodyside sensors on opposite sides (which we must), then we have

$$[\Gamma_1, \Gamma_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} \text{bodyside measurements} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} \text{roof measurements} \end{matrix}$$

which are orthogonal. Consequently, we cannot place the two roof sensors at equal distances from the left/right centerline of the liftgate opening, such as is the case with sensors 5 and 6 in Fig. 1(c). Otherwise, we would have

$$[\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & -c \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} \text{bodyside measurements} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} \text{roof measurements} \end{matrix}$$

where c is a constant that is proportional to the distance between the sensors and the left/right centerline of the liftgate opening. In this case, $[\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5]$ would constitute a set of $n = 4$ orthogonal pattern vectors, which violates condition (C6). In light of all of these constraints, we might consider placing the four sensors as shown in Fig. 6, for which we have

$$\Gamma = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{6}$$

It can be verified that the resulting Gram matrix is nonsingular, so that the layout shown in Fig. 6 is diagnosable.

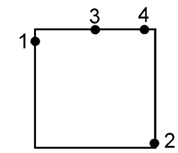

Fig. 6. Diagnosable layout with four sensors in the liftgate opening example.

C. Independent Versus Dependent Variation Sources

In the example of Section IV-A, it was possible to diagnose 15 variation patterns using only eight sensors. One of the reasons why it is often possible to find a diagnosable layout with $n < p$ is because of the assumption that the p variation sources are all independent. In this case, the covariance structure of $\mathbf{y}$ takes the form $\Sigma_y = \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \sigma_i^2 + \sigma^2 \mathbf{I}$, in which there are only $p+1$ unknowns $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2, \sigma^2\}$. We have $n(n+1)/2$ distinct elements of Σ_y with which to solve for the $p+1$ unknowns, however.

Although there are many situations in which the variation sources are independent, and virtually all of the variance components estimation algorithms were developed under this assumption, one may also be interested in the more general situation in which the variation sources are not independent. In this case, the covariance structure of $\mathbf{y}$ is $\Sigma_y = \sum_{j=1}^{p} \sum_{i=1}^{p} \Gamma_i \Gamma_j^T \mathrm{Cov}(u_i, u_j) + \sigma^2 \mathbf{I}$, where $\mathrm{Cov}(u_i, u_j)$ denotes the covariance between the ith and jth variation sources. There are $p(p+1)/2+1$ unknown variances or covariances for which we must solve, and we would require a minimum of $p+1$ sensors. Because of this, it may be preferable to use a version of the regular least squares algorithm in [4] for the case of dependent variation sources, rather than a version of the MLS algorithm.

V. Diagnosability for Alternative Sensor Noise Assumptions

It has been assumed throughout this paper that the sensors are homogeneous, and the common noise variance σ^2 is unknown and must be estimated. In this section, we briefly consider diagnosability conditions for different variants of this assumption. First, suppose the sensors are homogeneous, but that σ^2 is known or that a sufficiently accurate *a priori* estimate is available. For example, gage repeatability and reproducibility (gage R&R) studies are widely used to determine measurement systems capability in manufacturing by estimating measurement error variances [22]. Assuming we use a version of an MLS algorithm that estimates $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2\}$ by minimizing the (weighted or unweighted) sum of the squares of the elements of the error matrix $\mathbf{S}_y - \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \hat{\sigma}_i^2 - \sigma^2 \mathbf{I}$, the system is diagnosable iff the set $\{\Gamma_1 \Gamma_1^T, \Gamma_2 \Gamma_2^T, \ldots, \Gamma_p \Gamma_p^T\}$ is linearly independent. In other words, the upper left $p \times p$ block of the Gram matrix in (4) must be nonsingular. A repetition of the proof of Theorem 1 yields the following corollary, which holds regardless of n and p.

Corollary 1: If σ^2 is known for a homogeneous sensor system, then a necessary and sufficient condition for nondiagnosability is that conditions (C1) through (C3) all hold.

Now suppose that we have a nonhomogeneous sensor system, and denote the variance of the ith sensor noise by $\sigma_{v,i}^2 (i = 1, 2, \ldots, n)$. If the noise variances were known (e.g., through gage R&R studies), then we could estimate the variance components by minimizing the (weighted or unweighted) sum of the squares of the elements of $\mathbf{S}_y - \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \hat{\sigma}_i^2 - \mathrm{diag}\{\sigma_{v,1}^2, \sigma_{v,2}^2, \ldots, \sigma_{v,n}^2\}$. Because diagnosability in this case is identical to the case of homogeneous sensors with σ^2 known, Corollary 1 also applies to nonhomogeneous sensor systems with known noise variances.

If the noise variances are unknown, we can estimate the variance components and noise variances together by minimizing the (weighted or unweighted) sum of the squares of the elements of $\mathbf{S}_y - \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \hat{\sigma}_i^2 - \mathrm{diag}\{\hat{\sigma}_{v,1}^2, \hat{\sigma}_{v,2}^2, \ldots, \hat{\sigma}_{v,n}^2\}$. Define $\mathbf{e}_i$ to be the ith unit vector (an n-length column vector of zeros with a 1 as the ith element), so that this expression becomes $\mathbf{S}_y - \sum_{i=1}^{p} \Gamma_i \Gamma_i^T \hat{\sigma}_i^2 - \sum_{i=1}^{n} \mathbf{e}_i \mathbf{e}_i^T \hat{\sigma}_{v,i}^2$. It follows that the system is diagnosable iff the set $\{\Gamma_1\Gamma_1^T, \Gamma_2\Gamma_2^T, \ldots, \Gamma_p\Gamma_p^T, \mathbf{e}_1\mathbf{e}_1^T, \mathbf{e}_2\mathbf{e}_2^T, \ldots, \mathbf{e}_n\mathbf{e}_n^T\}$ is linearly independent. Once again, a straightforward repetition of the proof of Theorem 1 yields the following corollary, which holds regardless of n and p.

Corollary 2: If the noise variances are unknown for a nonhomogeneous sensor system, then a necessary and sufficient condition for nondiagnosability is that conditions (C1) through (C3) all hold with the set of p pattern vectors replaced by the augmented set $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $n + p$ vectors.

The conditions in Corollary 2 for nonhomogeneous sensor systems are much more restrictive than for homogeneous systems. We cannot have a variation pattern that affects only a single sensor, because the corresponding pattern vector would be collinear with one of the unit vectors. This would violate conditions (C1) and (C2) with $k = l = 1$. Similarly, we cannot have two variation patterns that affect the same two sensors but no other sensors, because the two corresponding pattern vectors would each be a linear combination of the same two unit vectors. This would violate conditions (C1) and (C2) with $k = l = 2$. In general, we should attempt to lay out the sensor system so that each variation pattern affects as many sensors as possible.

It is interesting to note that none of the examples considered in Section IV would be diagnosable if the sensor systems were nonhomogeneous with unknown noise variances. For the Γ matrix in (5), there are numerous violations of the above conditions. Patterns 4 and 5 affect only sensors 2 and 3, patterns 11 and 12 affect only sensors 5 and 6, pattern 13 affects only sensor 7, etc. For the Γ matrix in (6), there are similar violations of the above conditions.

VI. Conclusion

This paper has translated the diagnosability condition of Gram matrix singularity for a common variance component estimation algorithm into a set of conditions on the pattern vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$. It was shown that only certain types of linear dependencies among the pattern vectors result in nondiagnosability, whereas other types of linear dependencies are allowable. Because the pattern vectors have a much clearer physical interpretation than the Gram matrix, the characterization of diagnosability in terms of the pattern vectors can provide insight that is useful when laying out a sensor system. Examples from autobody panel assembly were used to illustrate the results.

The diagnosability characterization could also be used in conjunction with sensor layout optimization strategies (e.g., [23], [24]) that search over all possible candidate sensor layouts. Insight from the characterization could be used to reduce the search space by narrowing down the set of diagnosable candidate layouts or to provide an initial guess for the optimal layout.

As a final comment, we point out that all methods discussed in this paper are based on the assumption that variation sources have linear effects on the measurements, as represented by (1), and that the matrix Γ is known. This may be unrealistic for processes in which the effects of the variation sources are difficult to model and there is no historical database of common variation patterns. These situations require the more versatile but less powerful methods in which the variation pattern vectors and the variance components are simultaneously estimated, based on only the measurement data. Examples of such methods can be found in [1], [8], and [25].

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