# Supplement to: Optimal Maintenance Strategies for Wind Turbine Systems Under Stochastic Weather Conditions

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## Proofs

*Proof of Proposition 4*

Let $\pi_i^2$ and $(\pi P)_i$ denote the $i$th position of the row vector $\pi^2 (= \pi'(\pi))$ and $\pi P$, respectively. Then, we have

$$\sum_{i \geq j} \pi_i^2 \quad = \quad \sum_{i \geq j} \frac{(\pi P)_i}{R(\pi)} \leq \sum_{i \geq j} \frac{(\pi P)_i}{R(\hat{\pi})} \leq \sum_{i \geq j} \frac{(\hat{\pi} P)_i}{R(\hat{\pi})} = \sum_{i \geq j} \hat{\pi}_i^2 \tag{1}$$

The two inequalities in (1) hold due to Proposition 1(a) and Proposition 3(b), respectively.

*Proof of Proposition 5*

(a) We prove the claim by induction method. Without loss of generality, suppose that $CM_0(e_{m+1}) = C_{CM}$ and $PM_0(\pi) = C_{PM}$. Then, $CM_1(e_{m+1}) - C_{CM} = PM_n(\pi) - C_{PM} = \tau$. Suppose that $CM_n(e_{m+1}) - C_{CM} \geq PM_n(\pi) - C_{PM}$. Then,

$$CM_{n+1}(e_{m+1}) - C_{CM}$$

$$= (1 - W_{CM})(\tau + C_{CM} + V_n(e_1)) + W_{CM}(\tau + CM_n(e_{m+1})) - C_{CM} \tag{2}$$

$$= \tau + V_n(e_1) + W_{CM}(CM_n(e_{m+1})) - C_{CM} - V_n(e_1)) \tag{3}$$

$$\geq \tau + V_n(e_1) + W_{PM}(PM_n(e_{m+1})) - C_{PM} - V_n(e_1)) \tag{4}$$

$$= PM_{n+1}(\pi) - C_{PM}, \tag{5}$$

where (4) is from induction hypothesis. Therefore, $CM_n(e_{m+1}) - C_{CM} \geq PM_n(\pi) - C_{PM}$ holds for $\forall n$.

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(b)

$$
\begin{aligned}
CM_n(e_{m+1}) &= (1-W_{CM})(\tau+C_{CM}+V_{n-1}(e_1))+W_{CM}(\tau+CM_{n-1}(e_{m+1})), && (6)\\
&= \tau+C_{CM}+V_{n-1}(e_1)+W_{CM}(CM_{n-1}(e_{m+1})-C_{CM}-V_{n-1}(e_1)) && (7)\\
&\geq \tau+C_{PM}+V_{n-1}(e_1)+W_{PM}(PM_{n-1}(\tilde{\pi})-C_{PM}-V_{n-1}(e_1)) && (8)\\
&= PM_n(\pi). && (9)
\end{aligned}
$$

Inequality in (8) is due to the result of Proposition 5(a) and the fact that $C_{CM} \geq C_{PM}$ and $W_{CM} \geq W_{PM}$. Also, note that $PM_n(\tilde{\pi}) = PM_n(\pi)$. Consequently, $CM_n(e_{m+1}) \geq PM_n(\pi)$ for all $n$. ∎

*Proof of Lemma 1*

By induction, we can show that $V_n(\pi)$ is non-decreasing in $\prec_{st}$ when $P$ is IFR. Suppose that $n \geq T+1$ because one cannot carry out corrective maintenance when the system fails and the number of remaining periods is less than, or equal to, the lead time. Without loss of generality, we suppose that $V_{T+1}(\pi) = 0, \forall\pi$. Then, $NA_{T+2}(\pi) = (\tau T + CM_1(e_{m+1}))(1-R(\pi))$ is non-decreasing in $\prec_{st}$ from Proposition 3(a), and $PM_{T+2}(\pi)$ is constant in $\pi$. $OB_{T+2}(\pi) = C_{OB} + \sum_{i=1}^{m} min\{NA_{T+2}(e_i), PM_{T+2}(e_i)\}\pi_i$. Since $e_i \prec_{st} e_j$ for $i \leq j$ and $NA_{T+2}(e_i)$ is nondecreasing in $i$, $OB_{T+2}(\pi)$ is also non-decreasing in $\prec_{st}$ due to Proposition 2. Therefore, $V_{T+2}(\pi)$ is non-decreasing in $\prec_{st}$. Suppose that $V_n(\pi)$ is non-decreasing in $\prec_{st}$ for $\forall n \geq T+1$. Then, for $\pi \prec \hat{\pi}$,

$$
\begin{aligned}
NA_{n+1}(\pi) &= (\tau T+CM_{n-T}(e_{m+1}))(1-R(\pi))+V_n(\pi^2)R(\pi) && (10)\\
&\leq (\tau T+CM_{n-T}(e_{m+1}))(1-R(\pi))+V_n(\hat{\pi}^2)R(\pi) && (11)\\
&= (\tau T+CM_{n-T}(e_{m+1}))-(\tau T+CM_{n-T}(e_{m+1})-V_n(\hat{\pi}^2))R(\pi) && (12)\\
&\leq (\tau T+CM_{n-T}(e_{m+1}))-(\tau T+CM_{n-T}(e_{m+1}))-V_n(\hat{\pi}^2))R(\hat{\pi}) && (13)\\
&= (\tau T+CM_{n-T}(e_{m+1}))(1-R(\hat{\pi}))+V_n(\hat{\pi}^2)R(\hat{\pi}))=NA_{n+1}(\hat{\pi}) && (14)
\end{aligned}
$$

(11) follows from the induction assumption and Proposition 4. (13) follows from Proposition 3(a) and the fact that $\tau T + CM_{n-T}(e_{m+1}) \geq V_n(\pi), \forall\pi$ (Note that $\tau T + CM_{n-T}(e_{m+1})$ is the corrective maintenance costs *plus* revenue losses during the lead time when the system fails,

so it is always greater than the optimal value function for any operating state. Also, this can be easily proved mathematically by induction). It is obvious that $OB_{n+1}(\pi) = C_{OB} + \sum_i min\{NA_{n+1}(e_i), PM_{n+1}(e_i)\}\pi_i$ is also non-decreasing in $\prec_{st}$ with the similar reason explained above. Consequently, $V_{n+1}(\pi)$ is nondecreasing in $\prec_{st}$, $\forall n \geq T+1$.

Since $b(\pi)$ can be obtained by taking limits of $V_n(\pi)$, $b(\pi)$ is nondecreasing in $\prec_{st}$, which concludes the claim. ∎

*Proof of Lemma 2:*

$$b_{NA}(\pi) - b_{PM}(\pi) = C'_{CM}(1-R(\pi)) + b(\pi^2)R(\pi) - g - C'_{PM} \quad (15)$$

$$= (C'_{CM} - C'_{PM})(1-R(\pi)) - g + (b(\pi^2) - C'_{PM})R(\pi) \quad (16)$$

Note that $b(\pi^2) \leq C'_{PM}$. Consequently, if $(C'_{CM} - C'_{PM})(1-R(\pi)) - g \leq 0$ (or equivalently, $R(\pi) \geq 1 - \frac{g}{C'_{CM} - C'_{PM}}$), $NA$ is preferred to $PM$.

Next, Consider the case that $(C'_{CM} - C'_{PM})(1-R(\pi)) - g > 0$. Let us assume that $\delta^*(\pi) = NA$. Then,

$$b(\pi^2) - b(\pi) = b(\pi^2) - (C'_{CM}(1-R(\pi)) + b(\pi^2)R(\pi) - g) \quad (17)$$

$$= (b(\pi^2) - C'_{PM})(1-R(\pi)) - (C'_{CM} - C'_{PM})(1-R(\pi)) + g \quad (18)$$

(17) holds from the assumption $\delta^*(\pi) = NA$ and thus, $b(\pi) = C'_{CM}(1-R(\pi)) + b(\pi^2)R(\pi) - g$. Note that in (18), $b(\pi^2) \leq C'_{PM}$. Therefore, when $(C'_{CM} - C'_{PM})(1-R(\pi)) - g > 0$, $b(\pi^2) \leq b(\pi)$ with the assumption of $\delta^*(\pi) = NA$. But, this result contradicts that $b(\pi^2) \geq b(\pi)$ for $\pi \prec_{st} \pi'(\pi)$ from Lemma 1. Therefore, when $(C'_{CM} - C'_{PM})(1-R(\pi)) - g > 0$, or equivalently, $R(\pi) < 1 - \frac{g}{C'_{CM} - C'_{PM}}$, $NA$ cannot be optimal. ∎

*Proof of Theorem 1:* The first part is straightforward from Lemma 2 and the above discussions. Regarding the second part, $NA$ cannot be optimal at $\hat{\pi}$ from the fact that $R(\hat{\pi}) \leq R(\pi)$ for $\pi \prec_{st} \hat{\pi}$. Also, since $b(e_i)$ is non-decreasing in $i$, $\sum_i b(e_i)\pi_i$ is also non-decreasing in $\prec_{st}$-ordering from Proposition 2, and so is $b_{OB}(\pi)$. This leads to $b_{OB}(\hat{\pi}) \geq b_{OB}(\pi)$. But, $b_{PM}(\pi)$ is constant. Thus, when $\delta^*(\pi) = PM$, $OB$ cannot be optimal at $\hat{\pi}$ as well, which concludes the second part of the Theorem. ∎

*Proof of Corollary 1*

It follows directly from Lemma 2 and the fact that $OB$ is preferred to $PM$ when $C'_{PM} \geq C_{OB} + \sum b(e_i)\pi_i$. ∎

*Proof of Lemma 3:*

We use similar technique used in Lemma 2.

$$\begin{aligned} & b_{NA}(\pi) - b_{OB}(\pi) && (19)\\ & = C'_{CM}(1 - R(\pi)) + b(\pi^2)R(\pi) - g - C_{OB} - \sum b(e_i)\pi_i && (20)\\ & = (C'_{CM} - C_{OB} - \sum b(e_i)\pi_i)(1 - R(\pi)) - g + R(\pi)(b(\pi^2) - C_{OB} - \sum b(e_i)\pi_i) && (21)\\ & = (C'_{CM} - C_{OB} - \sum b(e_i)\pi_i)(1 - R(\pi)) - g + R(\pi)\sum b(e_i)(\pi_i^2 - \pi_i) \\ & + R(\pi)(b(\pi^2) - C_{OB} - \sum b(e_i)\pi_i^2), && (22) \end{aligned}$$

Note that $b(\pi^2) \leq C_{OB} + \sum b(e_i)\pi_i^2$. Therefore, if $(C'_{CM} - C_{OB} - \sum b(e_i)\pi_i)(1 - R(\pi)) - g + R(\pi)\sum b(e_i)(\pi_i^2 - \pi_i) \leq 0$, $b_{NA}(\pi) \leq b_{OB}(\pi)$. Re-arranging the condition yields

$$\begin{aligned} & (C'_{CM} - C_{OB} - \sum b(e_i)\pi_i)(1 - R(\pi)) - g + R(\pi)\sum b(e_i)(\pi_i^2 - \pi_i) < 0 && (23)\\ \Leftrightarrow \quad & R(\pi) \geq \frac{C'_{CM} - C_{OB} - \sum b(e_i)\pi_i - g}{C'_{CM} - C_{OB} - \sum b(e_i)\pi_i^2} && (24) \end{aligned}$$

The last inequality (24) comes from $b(e_i) \leq C'_{PM}$ for all $i = 1, \cdots, m$ and from $C_{OB} + C'_{PM} \leq C'_{CM}$ (Note that $C_{OB} + C_{PM} \leq C_{CM}$ by assumption). ∎

*Proof of Corollary 2*

It follows directly from Lemma 2 and Lemma 3. ∎

*Proof of Lemma 4:*

We will use contradiction. Assume that $\delta^*(\pi) = NA$. Then,

$$
\begin{aligned}
& b(\pi^2) - b(\pi) - \sum b(e_i)(\pi_i^2 - \pi_i) && (25)\\
= \;& b(\pi^2) - C'_{CM}(1 - R(\pi)) - b(\pi^2)R(\pi) + g - \sum b(e_i)(\pi_i^2 - \pi_i) && (26)\\
= \;& (b(\pi^2) - C'_{CM})(1 - R(\pi)) + g - \sum b(e_i)(\pi_i^2 - \pi_i) && (27)\\
= \;& (b(\pi^2) - C_{OB} - \sum b(e_i)\pi_i^2)(1 - R(\pi)) + (C_{OB} \\
& + \sum b(e_i)\pi_i - C'_{CM})(1 - R(\pi)) + g - R(\pi)\sum b(e_i)(\pi_i^2 - \pi_i) && (28)
\end{aligned}
$$

Note that $b(\pi^2) - C_{OB} - \sum b(e_i)\pi_i^2 \leq 0$. Also, by the condition of the claim, the remaining term is also negative. Therefore, we get $b(\pi^2) - b(\pi) - \sum b(e_i)(\pi_i^2 - \pi_i) < 0$ under the assumption of $\delta^*(\pi) = NA$. However,

$$
\begin{aligned}
& b(\pi^2) - b(\pi) - \sum b(e_i)(\pi_i^2 - \pi_i) && (29)\\
= \;& b_{OB}(\pi^2) - b(\pi) - b_{OB}(\pi^2) + b_{OB}(\pi) \quad \text{(from } \delta^*(\pi^2) = OB\text{)}, && (30)\\
= \;& -b(\pi) + b_{OB}(\pi) \geq 0, && (31)
\end{aligned}
$$

which contradicts the assumption. As a result, $\delta^*(\pi)$ cannot be $NA$. Also note that $b_{OB}(\pi) \leq b_{OB}(\pi^2) \leq b_{PM}(\pi^2) = b_{PM}(\pi)$. Therefore, $PM$ cannot be also optimal, which concludes $\delta^*(\pi) = OB$. ∎

*Proof of Corollary 3:*

(a) Applying Proposition 4 repeatedly to both sides of this inequality yields the result.

(b) Since the states along any sample path is in $\prec_{st}$-increasing order, the result follows directly from Lemma 1.

(c) Note that $R(\pi^k)$ is non-increasing in $k$ by proposition 3(a). Then, the result follows from Lemma 2.

(d) For $k \geq k1(\pi)$, $NA$ cannot be the optimal action from Lemma 2. Also, for $k \geq k2(\pi)$, $PM$ is preferable to $OB$ since $C_{OB} + \sum b(e_i)\pi_i^k$ is nondecreasing in $k$ in a $\prec_{st}$-increasing sample path and $C'_{PM}$ is constant. Hence for $k \geq k^*$, either $NA$ or $OB$ cannot be optimal. For $k1 \leq k < k^*$, $OB$ is optimal, whereas $k2 \leq k < k^*$, $NA$ is optimal. For $k < min\{k1, k2\}$, $OB$ or $NA$ is optimal. ∎

*Proof of Lemma 5:* We apply the similar induction technique used in [11]. Suppose that

$CM_0(e_{m+1}) = C_{CM}$. Also, suppose that $V_0(\pi) = 0$ for $\forall\pi$ for an operating system. $NA_1(\pi) = C_{CM}(1 - R(\pi))$ is linear in $\pi$. $OB_n(\pi)$ is hyperplane of $\pi$ and $PM_n(\pi)$ is constant in $\pi$ for $\forall n$. Therefore, $V_1(\pi)$ is piecewise linear concave because minimum of linear functions is piecewise linear concave. Now, suppose that $V_n(\pi)$ is piecewise linear concave such that $V_n(\pi) = min\{\pi \cdot a_n^T; a_n \in A_n\}$ where $a_n$ is a $1 \times (m+1)$ dimensional column vector. We only need to examine $NA_{n+1}(\pi)$ to show the piecewise linear concavity of $V_{n+1}(\pi)$. The first term of $NA_{n+1}(\pi)$, (that is, $(\tau T + CM_{n-T-1}(e_m + 1))(1 - R(\pi)))$ is linear in $\pi$. The second term of $NA_{n+1}(\pi)$ is,

$$R(\pi)V_n(\pi^2) = R(\pi)min\{\pi^2 \cdot a_n^T; a_n \in A_n\} \tag{32}$$

$$= R(\pi)min\left\{\left[\frac{(\pi P)_1}{R(\pi)}, \frac{(\pi P)_2}{R(\pi)}, \cdots, \frac{(\pi P)_m}{R(\pi)}, 0\right] \cdot a_n^T; a_n \in A_n\right\} \tag{33}$$

$$= min\{[(\pi P)_1, (\pi P)_2, \cdots, (\pi P)_m, 0] \cdot a_n^T; a_n \in A_n\} \tag{34}$$

$$= min\{\pi \cdot a_{n+1}^T; a_{n+1} \in A_{n+1}\} \tag{35}$$

Since $R(\pi)V_n(\pi^2)$ is the minimum of hyperplanes, it is piecewise linear concave, which makes $NA_{n+1}(\pi)$ is also piecewise linear concave. Consequently, $V_{n+1}(\pi)$ is piecewise linear concave. And the claim holds for $\forall n$ by induction. ∎

*Proof of Theorem 2:* Consider the two states $\pi(\lambda_1)$ and $\pi(\lambda_2)$ between $\pi$ and $\hat{\pi}$ ($\pi \prec_{st} \hat{\pi}$) where $\pi(\lambda_j) = \lambda_j\pi + (1-\lambda_j)\hat{\pi}$, for $j = 1, 2$ and $0 \le \lambda_1 \le \lambda_2 \le 1$. Then, from $\sum_{i\ge j}\pi_i \prec_{st} \lambda_1\sum_{i\ge j}\pi_i + (1-\lambda_1)\sum_{i\ge j}\hat{\pi}_i \prec_{st} \sum_{i\ge j}\hat{\pi}_i$, we have $\pi \prec_{st} \pi(\lambda_1) \prec_{st} \hat{\pi}$. In a similar way, we can easily show that $\pi(\lambda_1) \prec_{st} \pi(\lambda_2) \prec_{st} \hat{\pi}$. Therefore, $\pi(\lambda)$ is in $\prec_{st}$-increasing in $\lambda$, which implies that $b_{NA}(\pi(\lambda))$ and $b_{OB}(\pi(\lambda))$ is non-decreasing in $\lambda$. But, $b_{PM}(\pi(\lambda))$ is constant. Hence, there exists a control limit $\lambda^*$ such that for any $\lambda > \lambda^*$, $PM$ is optimal. The value of $\lambda^*$ is straightforward from Theorem 1. Next, let us consider $0 \le \lambda \le \lambda^*$. For this region, we already know that $PM$ cannot be optimal from Theorem 1. In Lemma 5, we show that $NA_n(\pi)$ is piecewise linear concave. Thus $b_{NA}(\pi)$ is also piecewise linear concave, but $b_{OB}(\pi)$ is hyperplane. Thus, $\{\pi; b_{NA}(\pi) \ge b_{OB}(\pi)\}$ is a convex set and thus, $\{\lambda; b_{NA}(\pi(\lambda)) \ge b_{OB}(\pi(\lambda)), 0 \le \lambda \le \lambda^*\}$ is also a convex set. This concludes the AM4R structure. ∎

*Proof of Corollary 4:*

When $\lambda_{NA\le PM} < \lambda_{OB\le PM}$, The second $NA$ region of AM4R structure vanishes. So the optimal policy structure results in at most three regions. The optimal policy region for $PM$ is straightforward from the previous discussions. $\blacksquare$